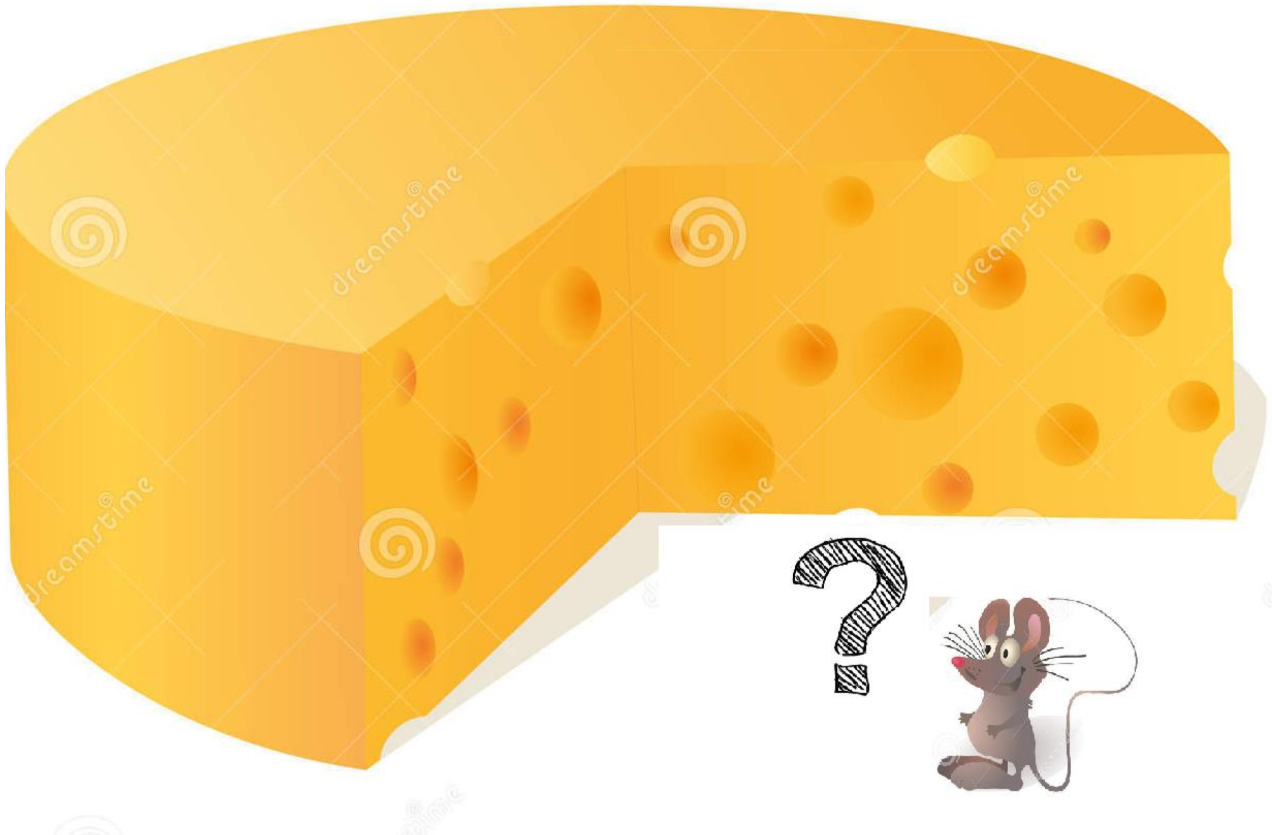
Big Data Class



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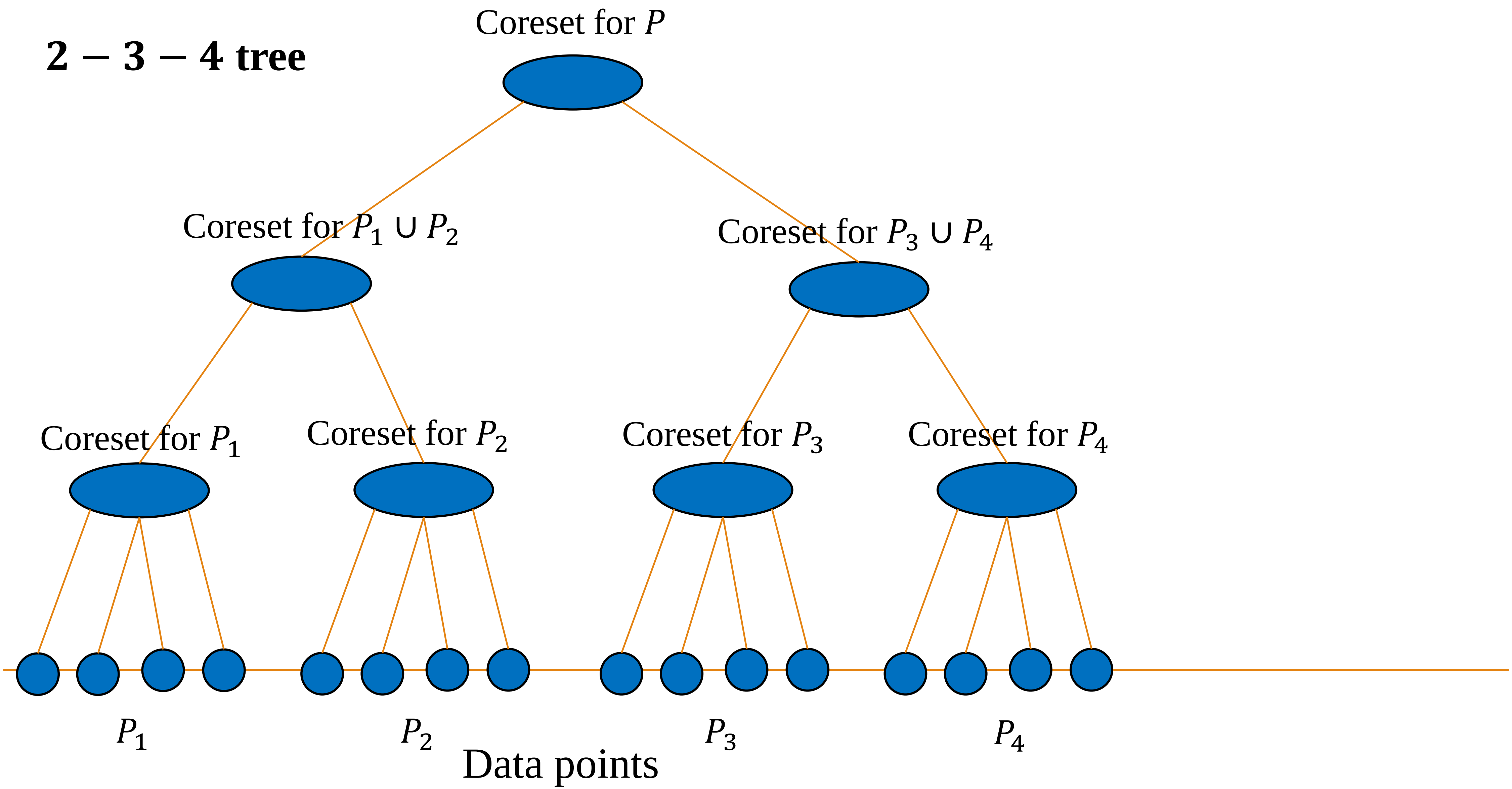
ALAA MAALOUF

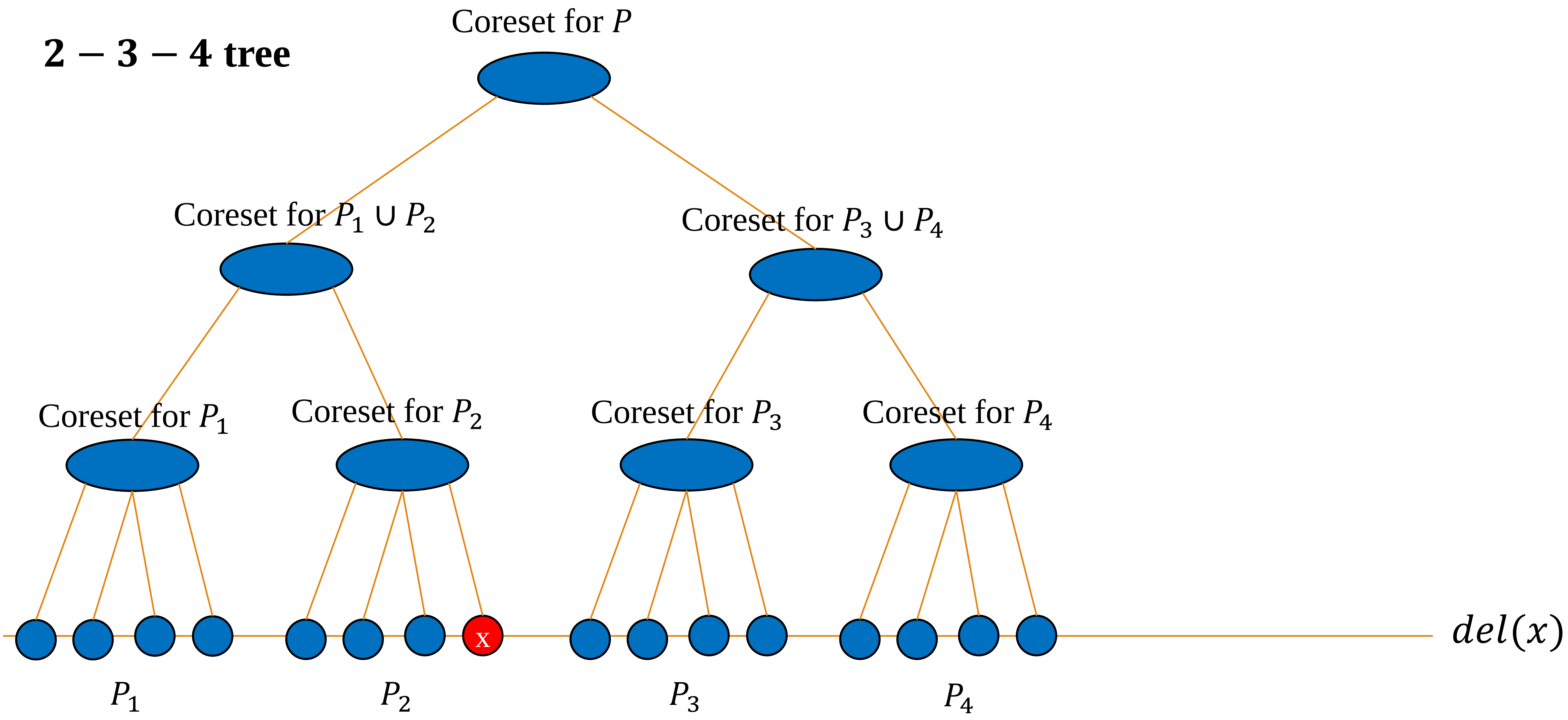
Dynamic Data with Insertions and Deletions

2 – 3 – 4 tree



Dynamic Data with Insertions and Deletions

2 − 3 − 4 tree



Dynamic Data with Insertions and Deletions

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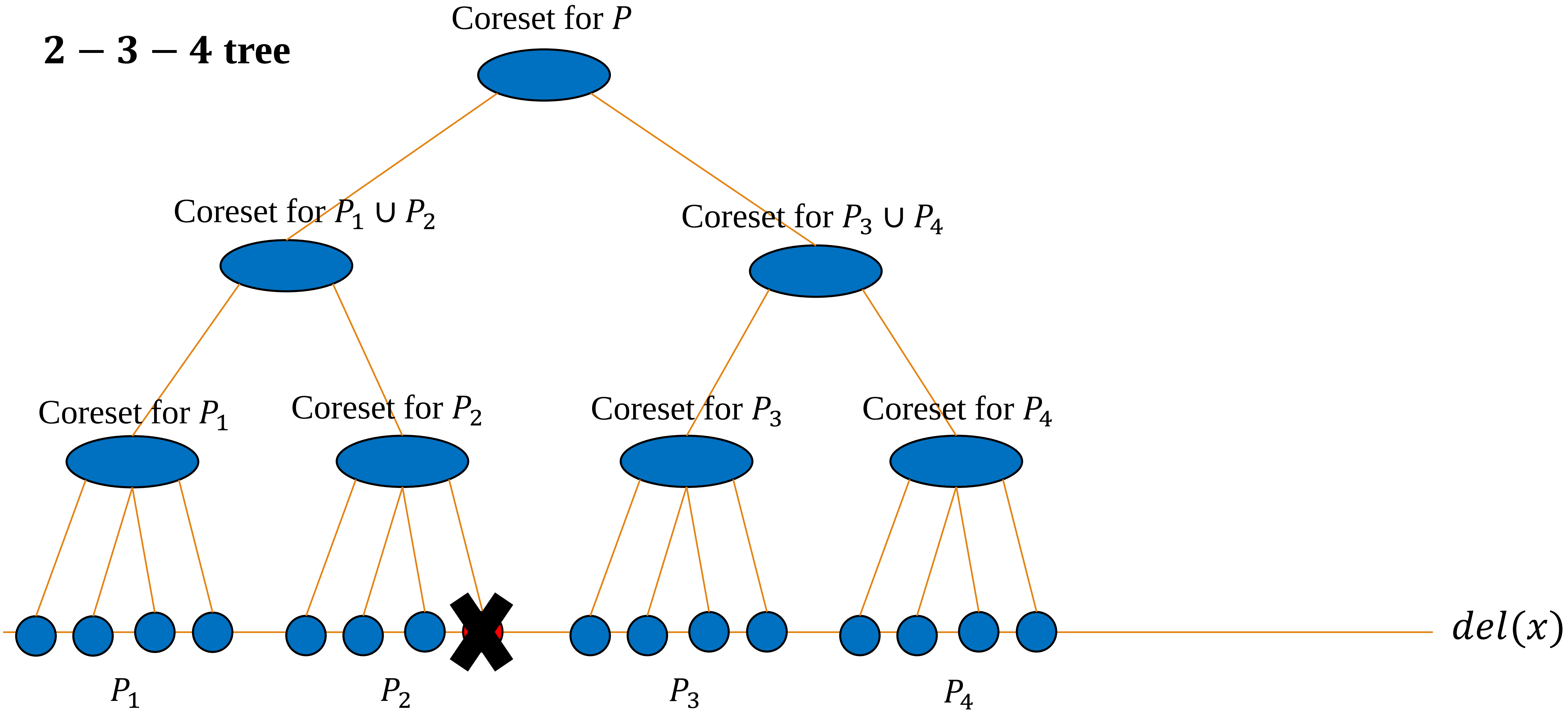

Dynamic Data with Insertions and Deletions

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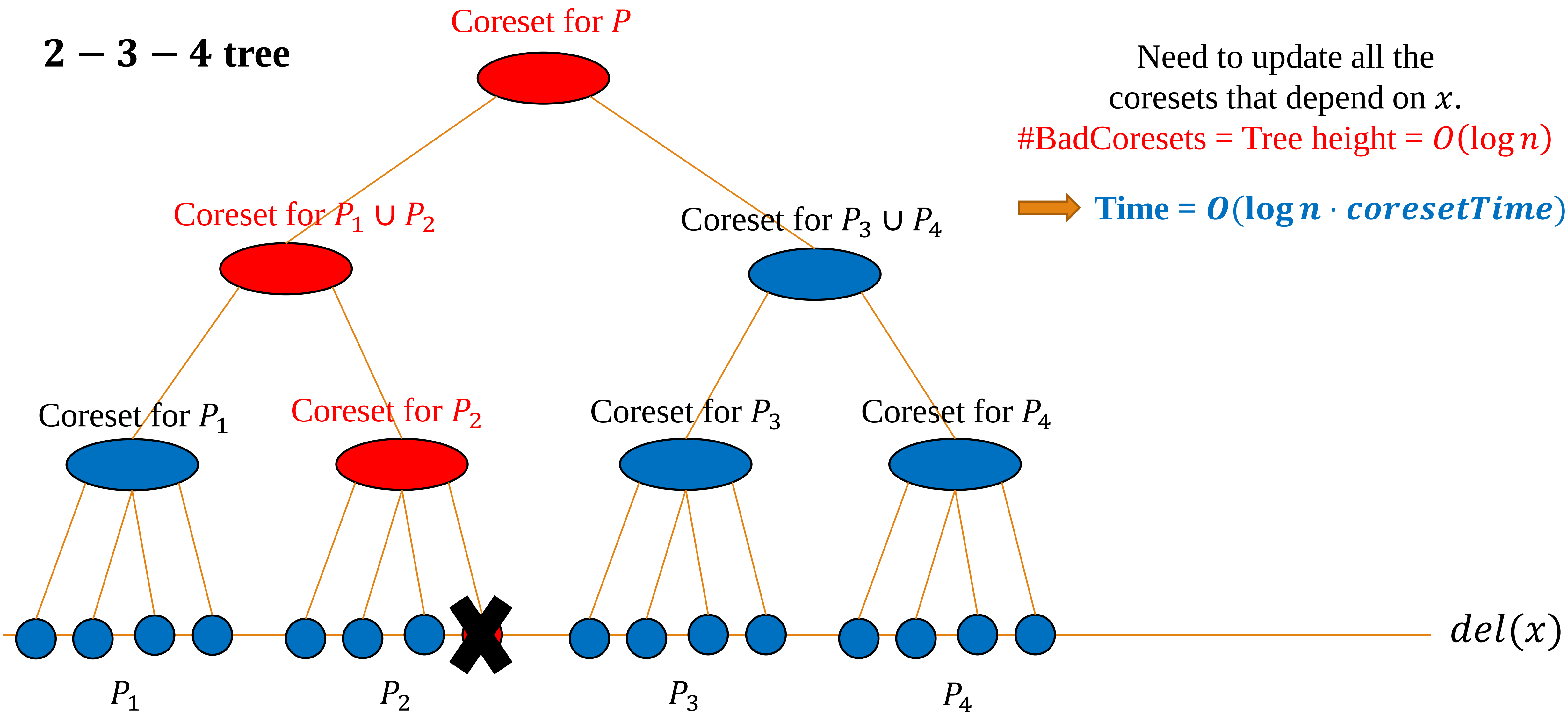
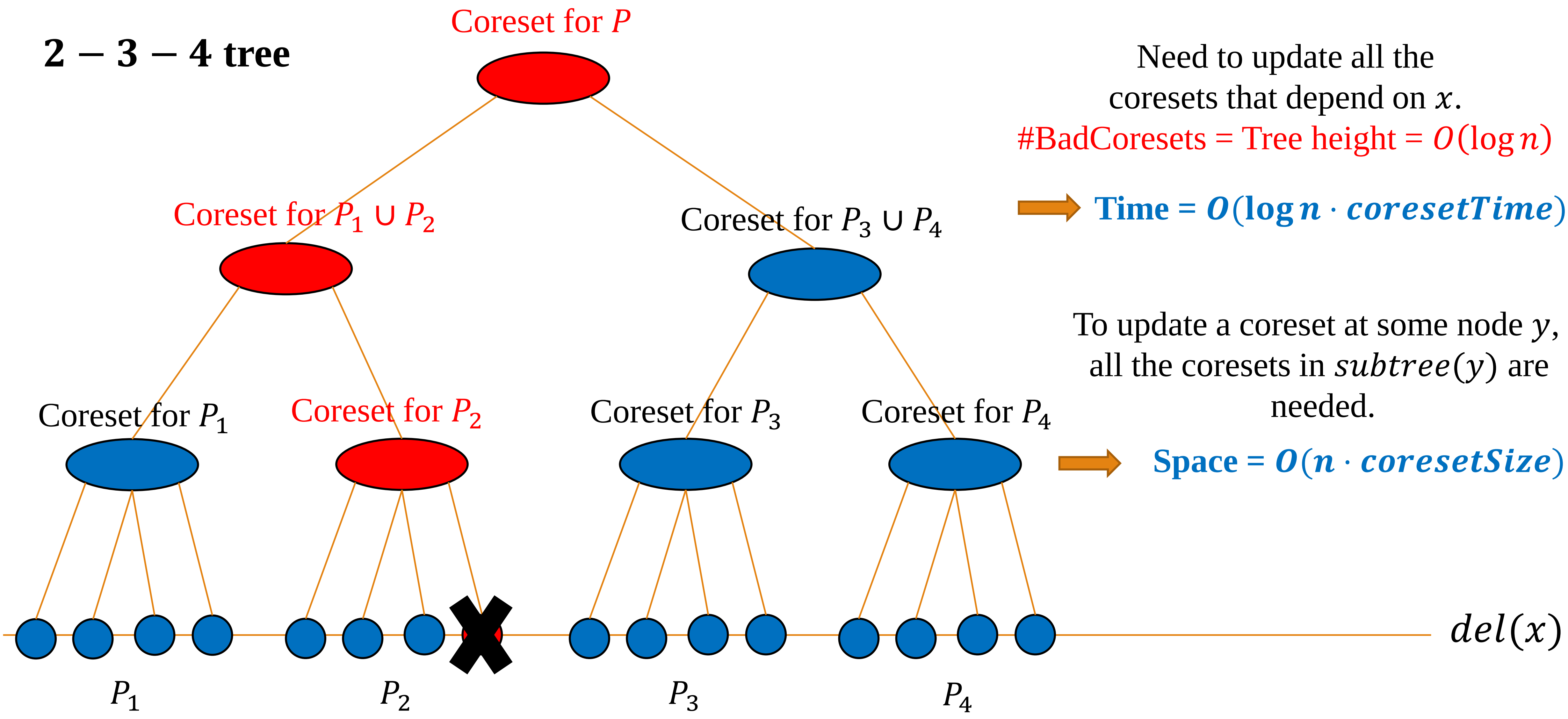


Need to update all the coresets that depend on x.

#BadCoresets = Tree height = $O(\log n)$

Time = $O(\log n \cdot coresetTime)$

Dynamic Data with Insertions and Deletions



Need to update all the coresets that depend on x.
#BadCoresets = Tree height = $O(\log n)$

⇒ **Time = $O(\log n \cdot coresetTime)$**

To update a coreset at some node y, all the coresets in $subtree(y)$ are needed.

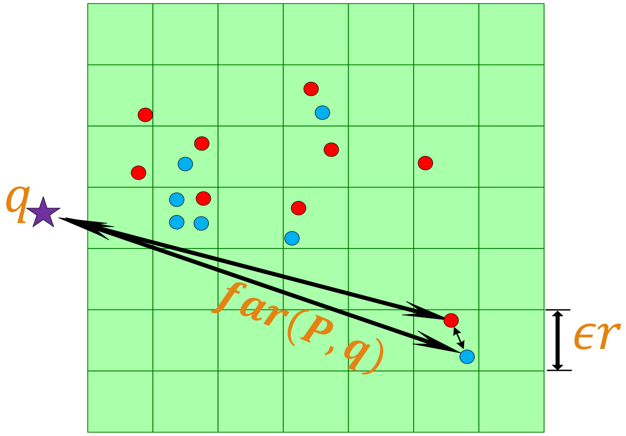
⇒ **Space = $O(n \cdot coresetSize)$**

Re-use of the *coreset* for 1-center

Reminder:

- We learned about coreset for $1 - center$.
- Given P, Q in R^d, such coreset $C \subseteq P$ guarantee that for every $q \in Q$:

$$\max_{p \in P} \|p - q\| \leq \max_{c \in C} \|c - q\| + O(\epsilon) \cdot \max_{p \in P} \|p - q\|$$



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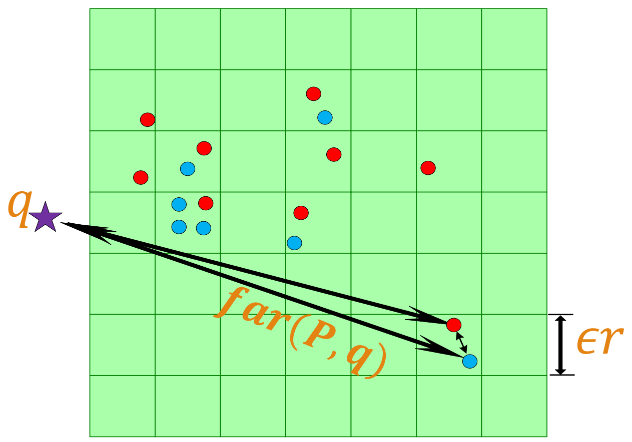

Question:

Can we use the same coreset for other problems/functions?

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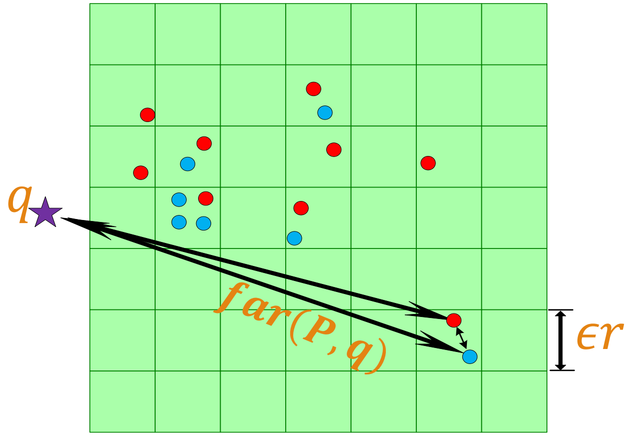

Question:

Can we use the same coreset for **Sum of distances ?**

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Can we use the same coreset for other problems/functions?

Re-use of the *coreset* for 1-center

Re

Question:

Can we use the same coreset for **Sum of squared distances ?**

…enter.

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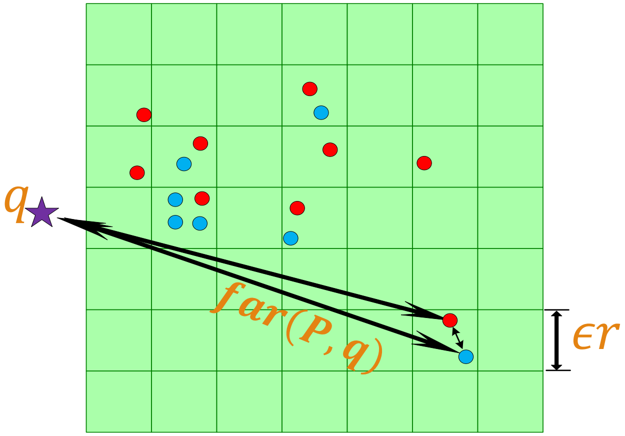

Question:

Can we use the same coreset for other problems/functions?

Re-use of $coreset\ for\ 1 - center$

Coreset for sum of distances ?

- Given coreset for $1 - center$, what error we get if we use it to measure sum, instead of max, of distances ?

- c_p := The representative of p
- $\forall c \in C, \quad \omega(c) := \left|\{p \in P \mid c_p = c\}\right|$

Claim:

$$\forall q \in Q: \quad \sum_{p \in P} \|p - q\| \le \sum_{c \in C} \omega(c)\|c - q\| + n \cdot O(\epsilon) \cdot \max_{p \in P} \|p - q\|$$

Re-use of the *coreset* for 1-center

Coreset for sum of distances - Proof

- $c_p :=$ The representative of p
- $\forall c \in C, \quad \omega(c) := \left|\{p \in P \mid c_p = c\}\right|$
- coreset for $1 - center$ implies that

$$\forall\, p \in P, q \in Q: \quad \|\boldsymbol{p} - \boldsymbol{q}\| \leq \left\|\boldsymbol{c_p} - \boldsymbol{q}\right\| + \boldsymbol{O}(\boldsymbol{\epsilon}) \cdot \max_{\boldsymbol{p} \in \boldsymbol{P}} \|\boldsymbol{p} - \boldsymbol{q}\|$$

Re-use of the *coreset* for 1-center

Coreset for sum of distances - Proof

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$$\forall q \in Q: \quad \sum_{\boldsymbol{p} \in \boldsymbol{P}} \|\boldsymbol{p} - \boldsymbol{q}\| \le \sum_{\boldsymbol{p} \in \boldsymbol{P}} \left(\left\|\boldsymbol{c_p} - \boldsymbol{q}\right\| + \boldsymbol{O}(\epsilon) \cdot \max_{\boldsymbol{p} \in \boldsymbol{P}} \|\boldsymbol{p} - \boldsymbol{q}\| \right)$$

$$= \sum_{\boldsymbol{p} \in \boldsymbol{P}} \left\|\boldsymbol{c_p} - \boldsymbol{q}\right\| + \boldsymbol{n} \cdot \boldsymbol{O}(\epsilon) \cdot \max_{\boldsymbol{p} \in \boldsymbol{P}} \|\boldsymbol{p} - \boldsymbol{q}\|$$

$$= \sum_{\boldsymbol{c} \in \boldsymbol{C}} \boldsymbol{\omega}(\boldsymbol{c}) \|\boldsymbol{c} - \boldsymbol{q}\| + \boldsymbol{n} \cdot \boldsymbol{O}(\epsilon) \cdot \max_{\boldsymbol{p} \in \boldsymbol{P}} \|\boldsymbol{p} - \boldsymbol{q}\|$$

Re-use of the *coreset* for 1-center

Coreset for sum of distances - Proof

- $c_p :=$ The representative of p
- $\forall c \in C, \quad \omega(c) := |\{p$ ...
- coreset for $1 - cent$ ...

$$\forall\, p \in P, q \in Q \; \ldots \; \max_{p \in P} \|p - q\|$$

Question:

Can we use the same coreset for **Sum of distances ?**

$$\forall q \in Q: \quad \sum_{p \in P} \|p - q\| \le \sum_{p \in P} \left(\|c_p - q\| + O(\epsilon) \cdot \max_{p \in P} \|p - q\| \right)$$

$$= \sum_{p \in P} \|c_p - q\| + n \cdot O(\epsilon) \cdot \max_{p \in P} \|p - q\|$$

$$= \sum_{c \in C} \omega(c) \|c - q\| + n \cdot O(\epsilon) \cdot \max_{p \in P} \|p - q\|$$

Re-use of the *coreset* for 1-center

Coreset for sum of squared distances ?

- Given coreset for $1 - center$, what error we get if we use it to measure sum of squared, instead of max, distances ?

- c_p ≔ The representative of p

- Let's look at the error:

$$\forall\ p \in P, q \in Q: \quad \left| \|\boldsymbol{p} - \boldsymbol{q}\|^2 \ - \ \left\|\boldsymbol{c_p} - \boldsymbol{q}\right\|^2 \right|$$

$$= \left| \|\boldsymbol{p}\|^2 + \|\boldsymbol{q}\|^2 - 2 \cdot \boldsymbol{p}^T\boldsymbol{q} - \left\|\boldsymbol{c_p}\right\|^2 - \|\boldsymbol{q}\|^2 + 2 \cdot \boldsymbol{c}_p^T\boldsymbol{q} \right|$$

$$\leq \left| \|\boldsymbol{p}\|^2 - \left\|\boldsymbol{c_p}\right\|^2 + 2 \cdot (\boldsymbol{c_p} - \boldsymbol{p})^T \boldsymbol{q} \right|$$

$$\leq \ ??$$

Re-use of the *coreset* for 1-center

Coreset for sum of squared distances ?

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$$\leq \left| \|\boldsymbol{p}\|^2 - \left\|\boldsymbol{c_p}\right\|^2 + 2 \cdot \left(\boldsymbol{c_p} - \boldsymbol{p}\right)^T \boldsymbol{q} \right|$$

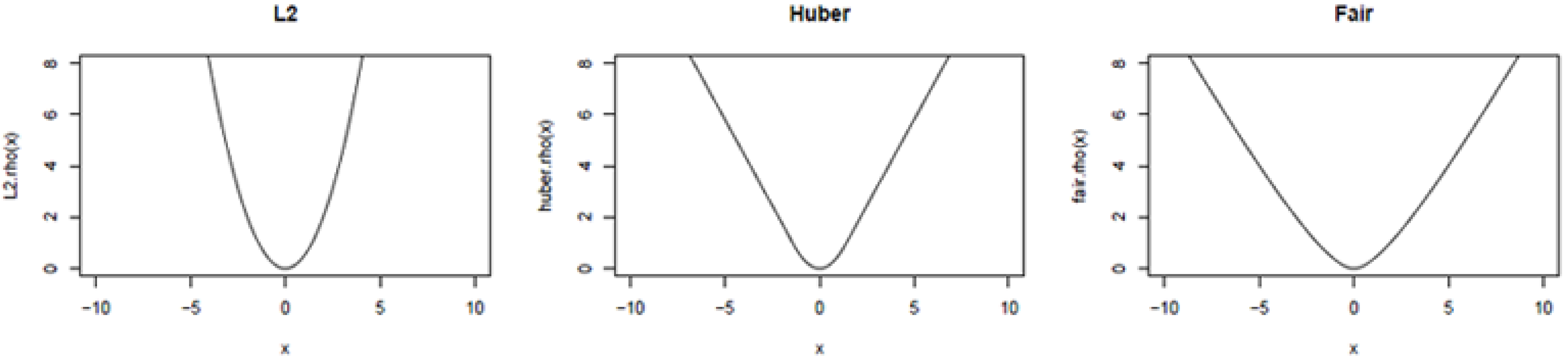
$$\leq \ ??$$

Question:

Can we use the same coreset for **Sum of squared distances ?**

Motivation

M-estimators:



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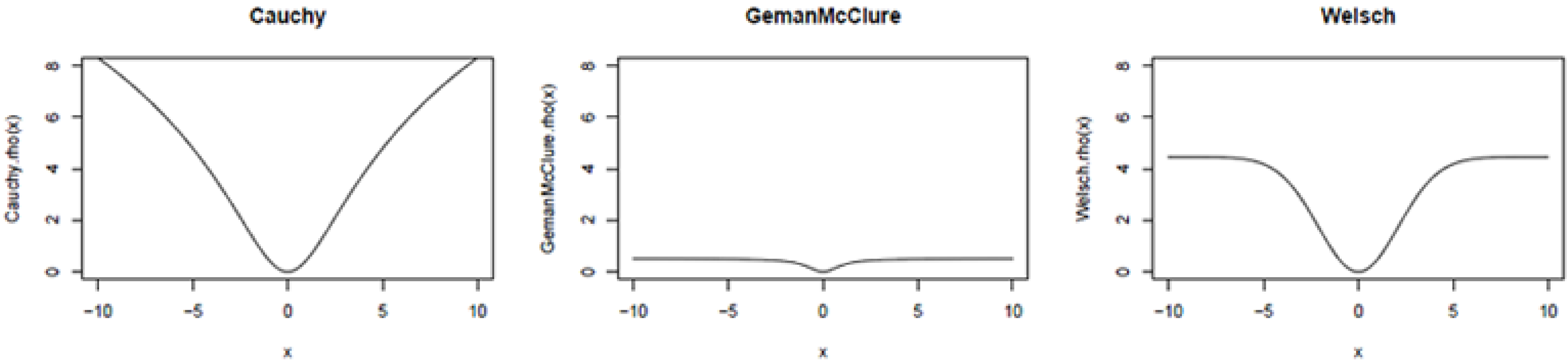
M-estimators:



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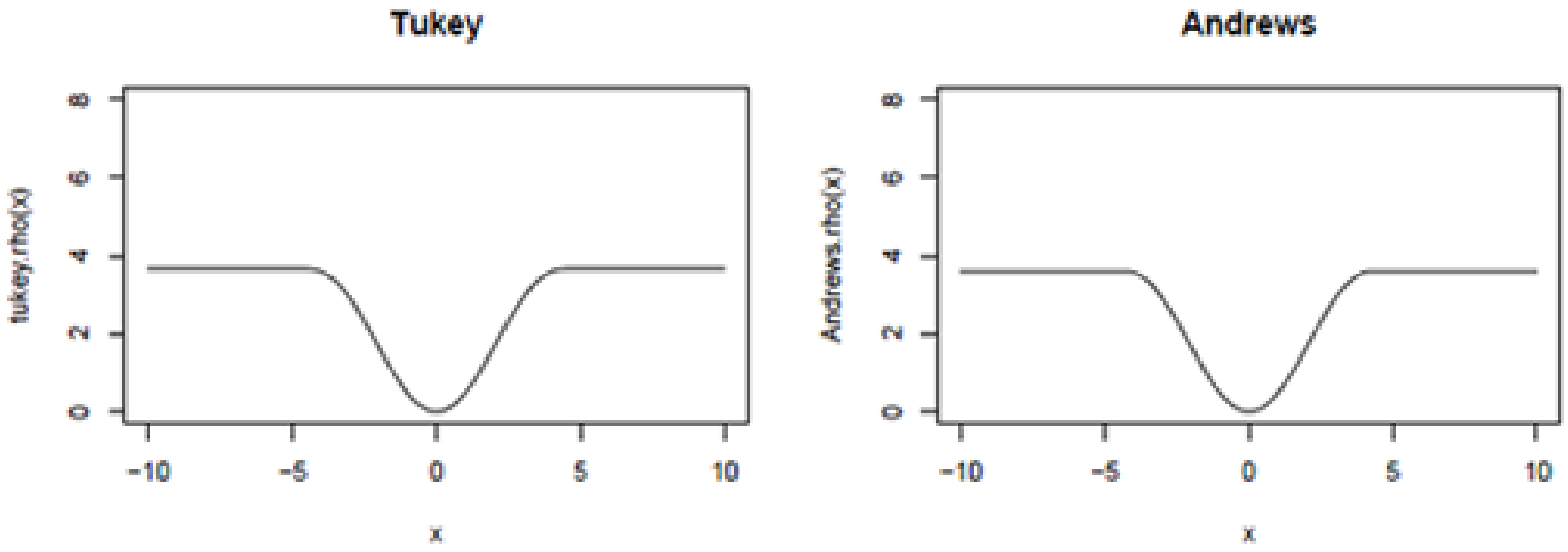


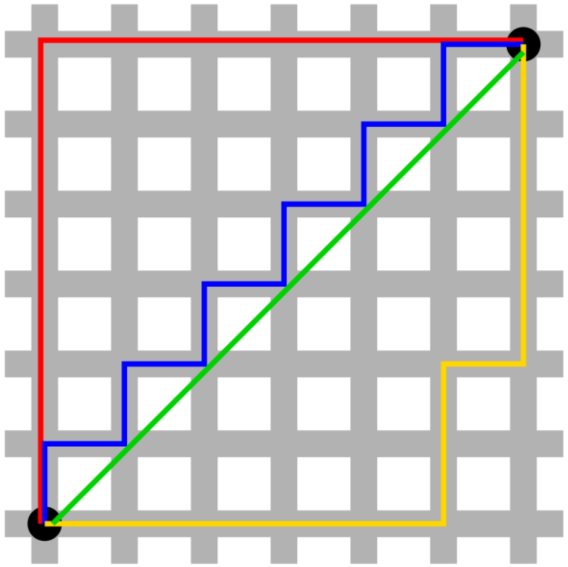
Distance Function (Metric)

- A distance function is a function that defines a distance between each pair of elements of a set X

$$d: X \times X \rightarrow [0, \infty)$$

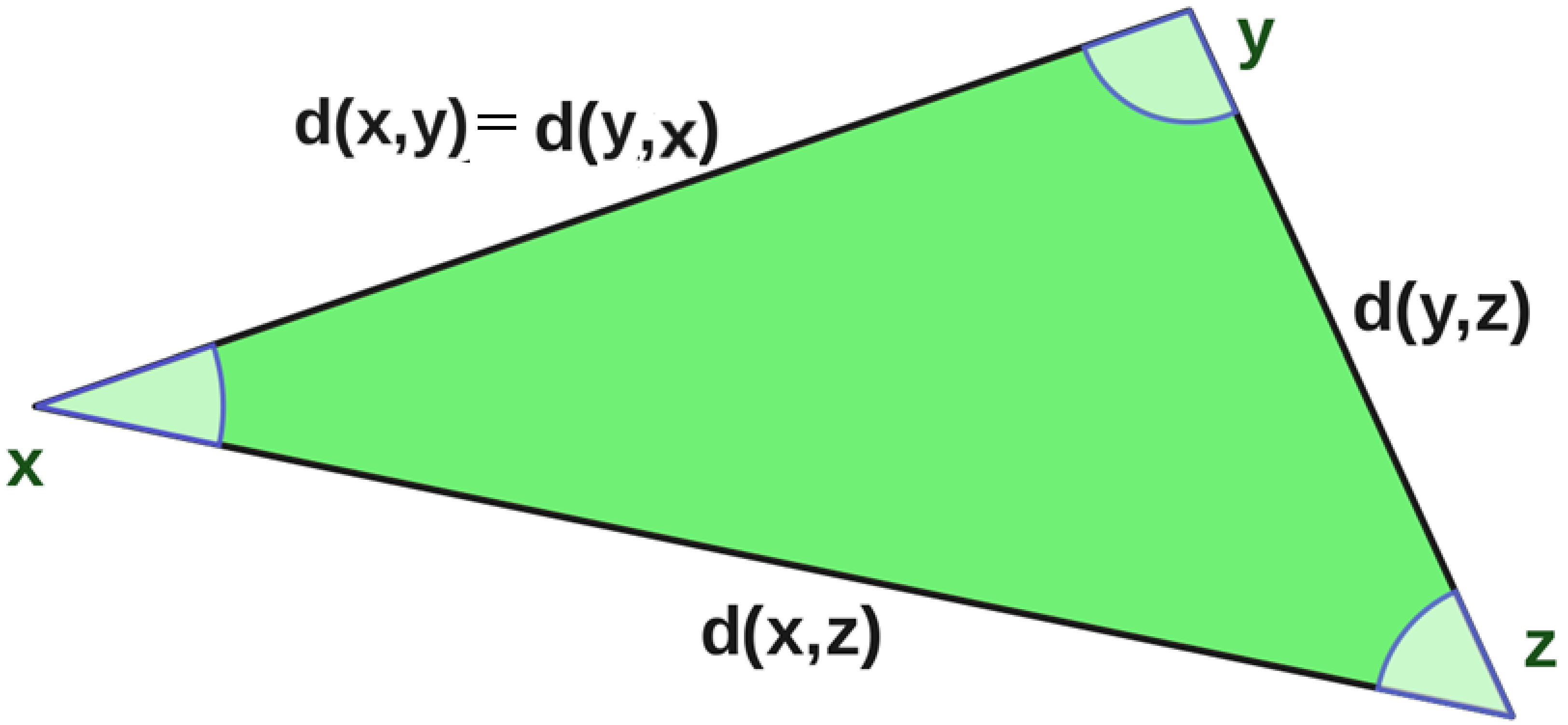
- A distance function satisfies the following conditions for every $x, y, z \in X$:

1) $d(x,y) \geq 0$ and $d(x,y) = 0 \leftrightarrow x = y$ non-negativity axiom.
2) $d(x,y) = d(y,x)$ symmetry.
3) $d(x,z) \leq d(x,y) + d(y,z)$ triangle-inequality.



Distance Function (Metric)

- Illustration:



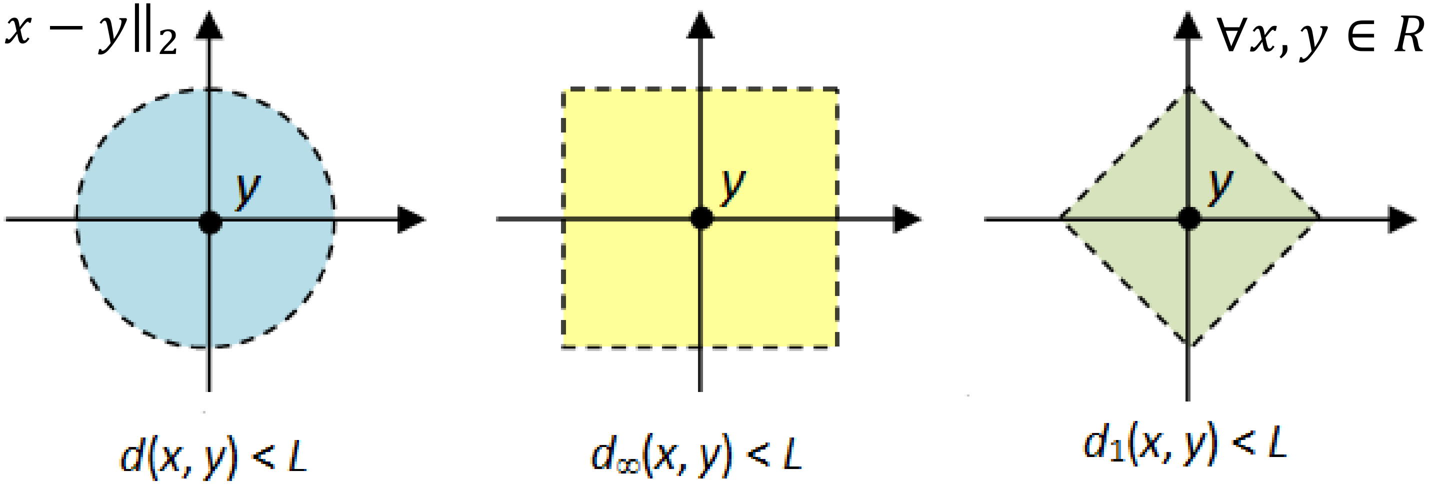
Metric Space

- A metric space is an ordered pair (X, d) where X is a set and $d: \mathrm{X} \times X \to [0, \infty)$ is a distance function on the set X.

Metric space (R, d_∞),
$\forall x, y \in R^2: d_\infty(x, y) = \|x - y\|_\infty$

Metric space (R, d),
$\forall x, y \in R^2: d(x, y) = \|x - y\|_2$

Metric space (R, d_1),
$\forall x, y \in R^2: d_1(x, y) = \|x - y\|_1$



Definitions

1) *Log-Log* Lipschitz:

A monotonic non-decreasing function $D: [0, \infty) \to [0, \infty)$ satisfies the *Log-Log* Lipschitz condition if there is $r > 0$ such that for every $x > 0$ and $\Delta > 1$ it holds that:

$$D(\Delta x) \leq \Delta^r D(x)$$

Lipschitz constant

2) Weak triangle inequality:

Let $D: X \times X \to [0, \infty)$. The function D satisfies the weak triangle inequality if there is $\rho > 0$ such that for every $(p, p', c) \in X^3$ the following holds:

$$D(p, c) \leq \rho\big(D(p, p') + D(p', c)\big)$$

Definitions

3) Property 1:

Let $D: X \times X \to [0, \infty)$. For every $\psi \in \left(0, \frac{1}{2}\right)$ there is a real $\phi \geq 0$ such that for every $(p, p', c) \in X^3$ it holds that

$$|D(p,c) - D(p',c)| \leq \phi D(p,p') + \psi D(p,c)$$

Main Claims

Let $(X, dist)$ be a metric space and let $f: [0, \infty) \to [0, \infty)$ be a function that satisfies the Log-Log Lipschitz condition.

Define $far: X^2 \to [0, \infty)$ to be a mapping from every $p, c \in X$ to

$$far(p, c) = f(dist(p, c))$$

- **Claim 1:**

The function far satisfies the weak triangle inequality for $\rho = \max\{2^{r-1}, 1\}$, i.e, for every $p, q, c \in X^3$:

$$far(p, q) \leq \rho\big(far(p, c) + far(c, q)\big)$$

Main Claims

- **Proof of Claim 1:**

Let $x = dist(p, c),\ y = dist(c, q),\ z = dist(p, q)$. Need to prove for $\rho = \max\{2^{r-1}, 1\}$ it holds that:

$$far(p, q) = \boldsymbol{f(z)} \leq \boldsymbol{\rho} \cdot \left(\boldsymbol{f(x)} + \boldsymbol{f(y)}\right) = \rho \cdot \left(far(p, c) + far(c, q)\right)$$

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By the triangle inequality and the fact that $f(\Delta \mathrm{x}) \le \Delta^r f(x)$, for every $\omega \in (0,1)$ it holds that:

$$f(z) \le f(x + y) = \omega \cdot f(x + y) + (1 - \omega) \cdot f(x + y)$$

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$$\le \omega \cdot f\left(\frac{x(x+y)}{x}\right) + (1-\omega) \cdot f\left(\frac{y(x+y)}{y}\right)$$

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Log-Log Lipschitz

$$\leq \omega \cdot f(x)\left(\frac{x+y}{x}\right)^r + (1-\omega) f(y)\left(\frac{x+y}{y}\right)^r$$

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Log-Log Lipschitz

$$\le \omega \cdot f(x)\left(\frac{x+y}{x}\right)^r + (1-\omega) f(y)\left(\frac{x+y}{y}\right)^r$$

$$= (x+y)^r \left(\frac{\omega \cdot f(x)}{x^r} + \frac{(1-\omega) f(y)}{y^r}\right)$$

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Let $x = dist(p,c),\ y = dist(c,q),\ z = dist(p,q)$. Need to prove for $\rho = \max\{2^{r-1}, 1\}$ it holds that:

$$far(p,q) = \boldsymbol{f(z) \leq \rho \cdot (f(x) + f(y))} = \rho \cdot (far(p,c) + far(c,q))$$

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$$= (x+y)^r\left(\frac{\omega \cdot f(x)}{x^r} + \frac{(1-\omega) f(y)}{y^r}\right)$$

By substituting $\omega = \frac{x^r}{x^r+y^r}$

ρ

$$= \left(\frac{(x+y)^r}{x^r+y^r}\right) \cdot \left(f(x) + f(y)\right)$$

Main Claims

Let $(X, dist)$ be a metric space and let $f: [0, \infty) \to [0, \infty)$ be a function that satisfies the Log-Log Lipschitz condition.

Define $far: X^2 \to [0, \infty)$ to be a mapping from every $p, c \in X$ to

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- **Claim 2:**

The function far satisfies property 1 for every $\psi \in \left(0, \frac{1}{2}\right)$ and $\phi = \max\left\{\left(\frac{r}{\psi}\right)^r, 1\right\}$

Main Claims

- **Proof of Claim 2:**

Let $x = dist(p,c),\ y = dist(c,q),\ z = dist(p,q), \psi \in \left(0, \frac{1}{2}\right)$ and $\phi = \left(\frac{r}{\psi}\right)^r$.

Need to prove that:

$$far(p,c) - far(c,q) = \boldsymbol{f(x) - f(y) \leq \phi f(z) + \psi f(x)} = \phi far(p,q) + \psi far(p,c)$$

Assume $f(x) > \phi f(z)$, otherwise Claim 2 holds trivially.

Main Claims

- **Proof of Claim 2:**

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Assume $f(x) > \phi f(z)$, otherwise Claim 2 holds trivially.

By the Log-Log Lipschitz it holds that $f(x) = f\left(y \cdot \frac{x}{y}\right) \leq \left(\frac{x}{y}\right)^r \cdot f(y)$

Main Claims

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By the Log-Log Lipschitz it holds that $f(x) = f\left(y \cdot \frac{x}{y}\right) \leq \left(\frac{x}{y}\right)^r \cdot f(y)$

$$\rightarrow f(y) \geq f(x) \cdot \left(\frac{y}{x}\right)^r$$

Main Claims

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Let $x = dist(p,c),\ y = dist(c,q),\ z = dist(p,q), \psi \in \left(0, \frac{1}{2}\right)$ and $\phi = \left(\frac{r}{\psi}\right)^r$.

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By the Log-Log Lipschitz it holds that $f(x) = f\left(y \cdot \frac{x}{y}\right) \leq \left(\frac{x}{y}\right)^r \cdot f(y)$

$$\rightarrow f(y) \geq f(x) \cdot \left(\frac{y}{x}\right)^r$$

Hence, $f(x) - f(y) \leq f(x) \cdot \left(1 - \left(\frac{y}{x}\right)^r\right)$.

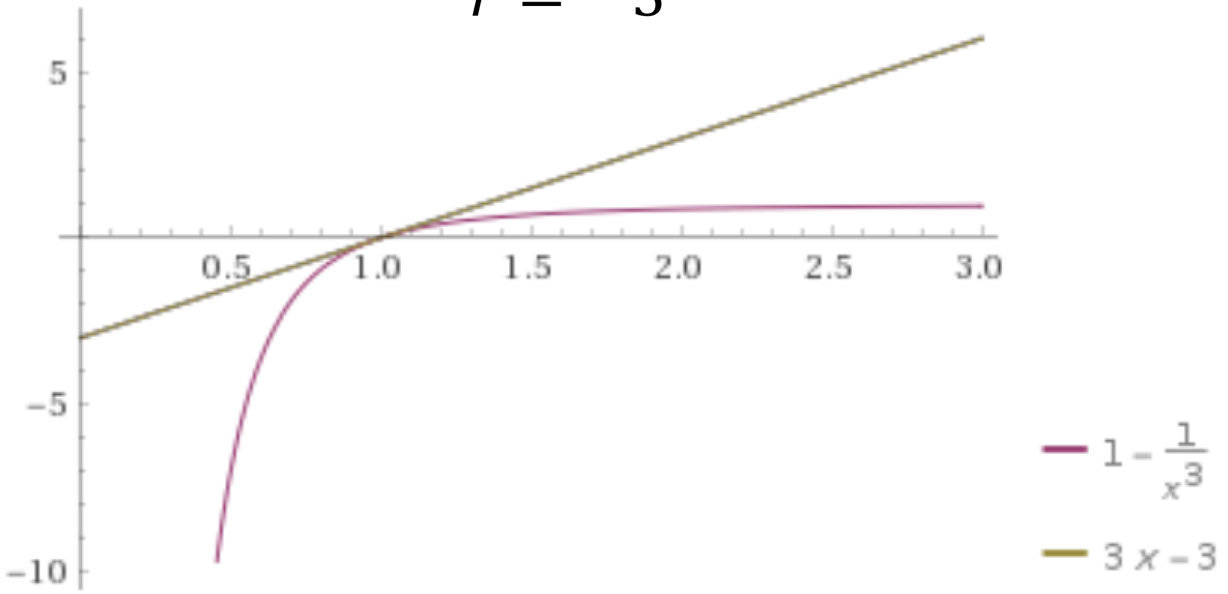
Main Claims

- **Proof of Claim 2:**

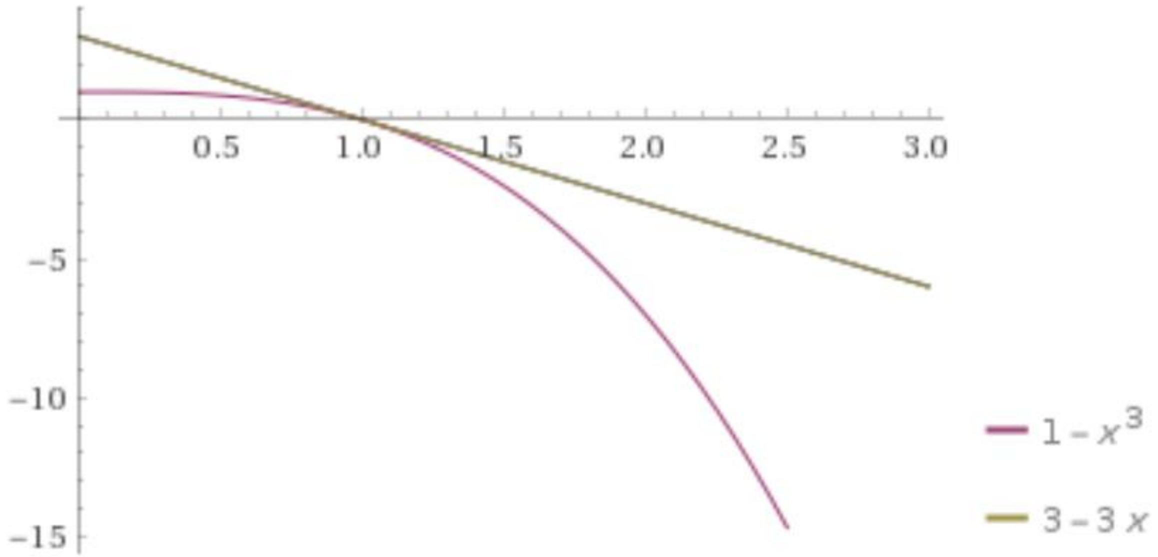
Hence, $f(x) - f(y) \leq f(x) \cdot \left(1 - \left(\frac{y}{x}\right)^r\right)$.

For every $\omega \geq 0$ it holds that $1 - \omega^r \leq r(1 - \omega)$.

$r = -3$



$r = 3$



Main Claims

- **Proof of Claim 2:**

Hence, $f(x) - f(y) \leq f(x) \cdot \left(1 - \left(\frac{y}{x}\right)^r\right)$.

For every $\omega \geq 0$ it holds that $1 - \omega^r \leq r(1 - \omega)$.

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Main Claims

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Triangle inequality

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$$\leq f(x) \cdot r\phi^{\frac{1}{r}}$$

Assumption:

$\phi f(z) < f(x)$

$\rightarrow f\left(\frac{z}{\phi^{\frac{1}{r}}}\right) \leq \phi f(z) < f(x)$

$\rightarrow \frac{z}{\phi^{\frac{1}{r}}} \leq z < x$ since f is non-decreasing

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$$= f(x) \cdot r\left(\frac{x - y}{x}\right) \leq f(x) \cdot r\left(\frac{z}{x}\right)$$

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Since $\phi = \left(\frac{r}{\psi}\right)^r$

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$$\leq f(x) \cdot r\phi^{\frac{1}{r}} \leq \psi f(x) \leq \boldsymbol{\phi f(z) + \psi f(x)}$$

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Main Claims

Let $(X, dist)$ be a metric space and let $f: [0, \infty) \to [0, \infty)$ be a function that satisfies the Log-Log Lipschitz condition.

Define $far: X^2 \to [0, \infty)$ to be a mapping from every $p, c \in X$ to

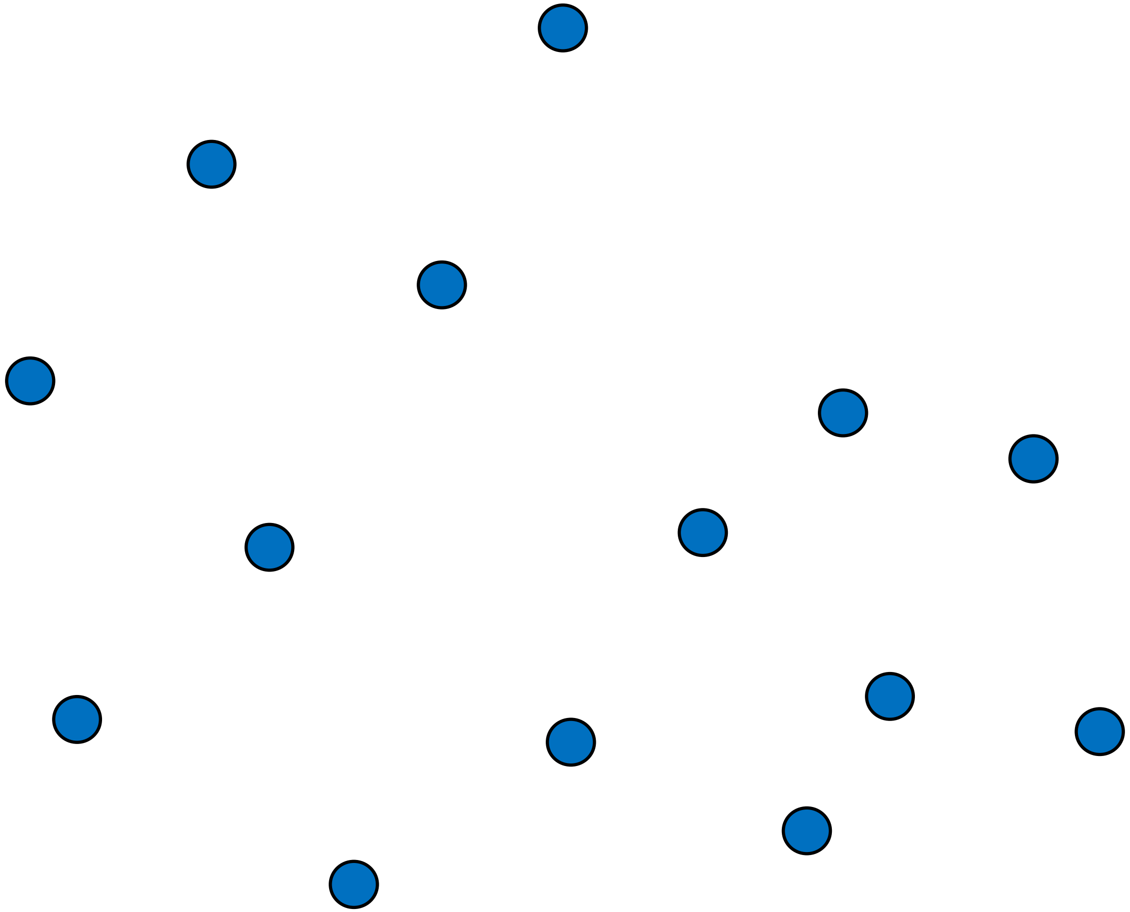
$$far(p, c) = f(dist(p, c))$$

- **Claim 3:** (Tighter bound than in Claim 2)

The function far satisfies property 1 for every $\psi \in \left(0, \frac{1}{2}\right)$ and $\phi = \max\left\{\left(\frac{r-1}{\psi}\right)^{r-1}, 1\right\}$ if $r > 1$.

General Bi-creteria Algorithm

$\underline{\boldsymbol{BICRETERIA(P, dist, \epsilon, \alpha, \beta):}}$



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***BICRETERIA*$(P, dist, \epsilon, \alpha, \beta)$:**

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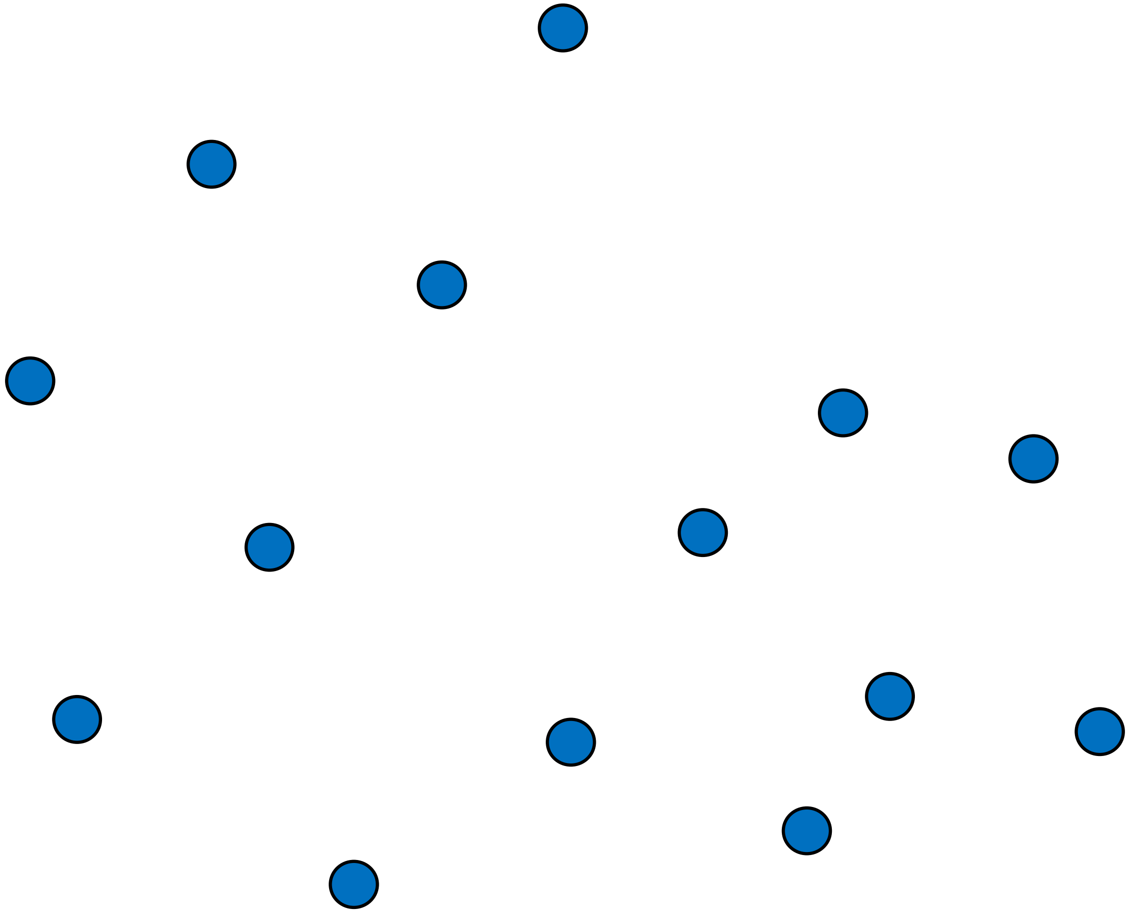

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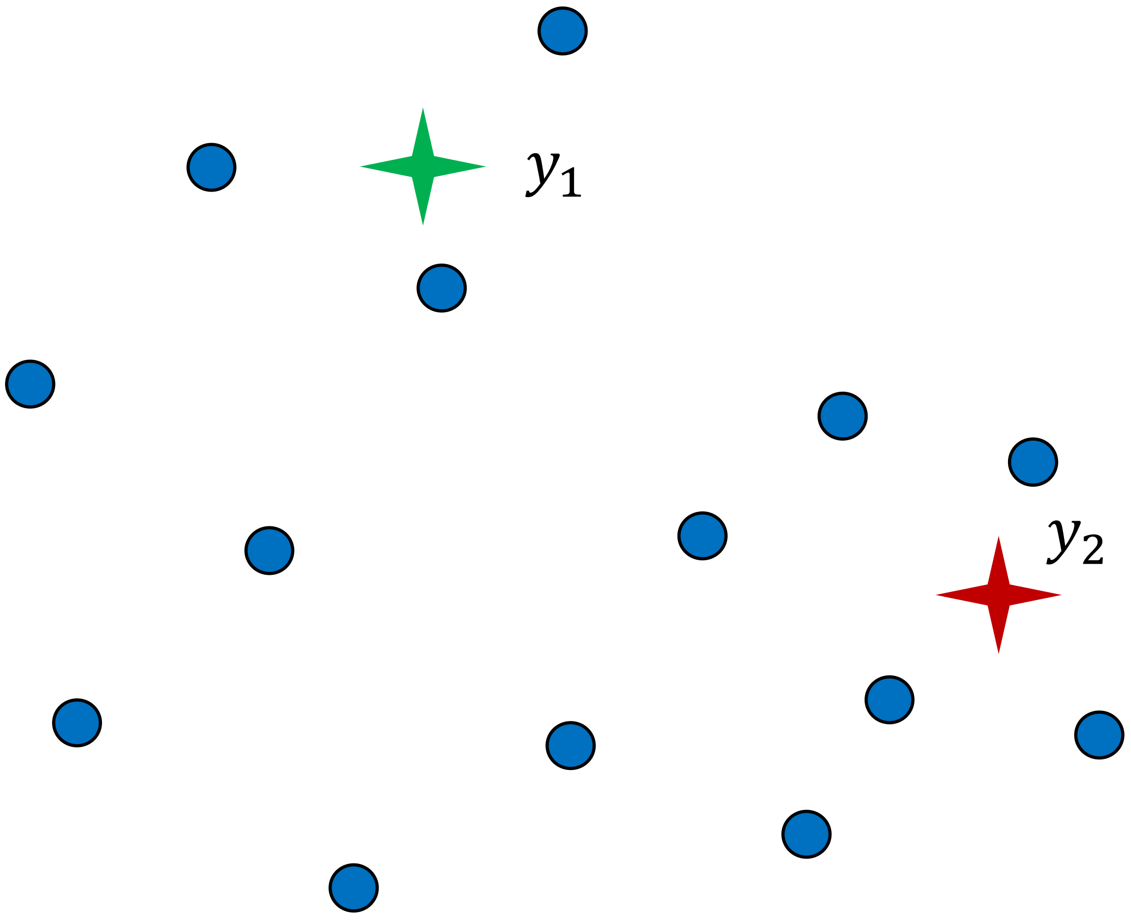
$\boldsymbol{Y_1 = \{y_1, \dots, y_\beta\}}$



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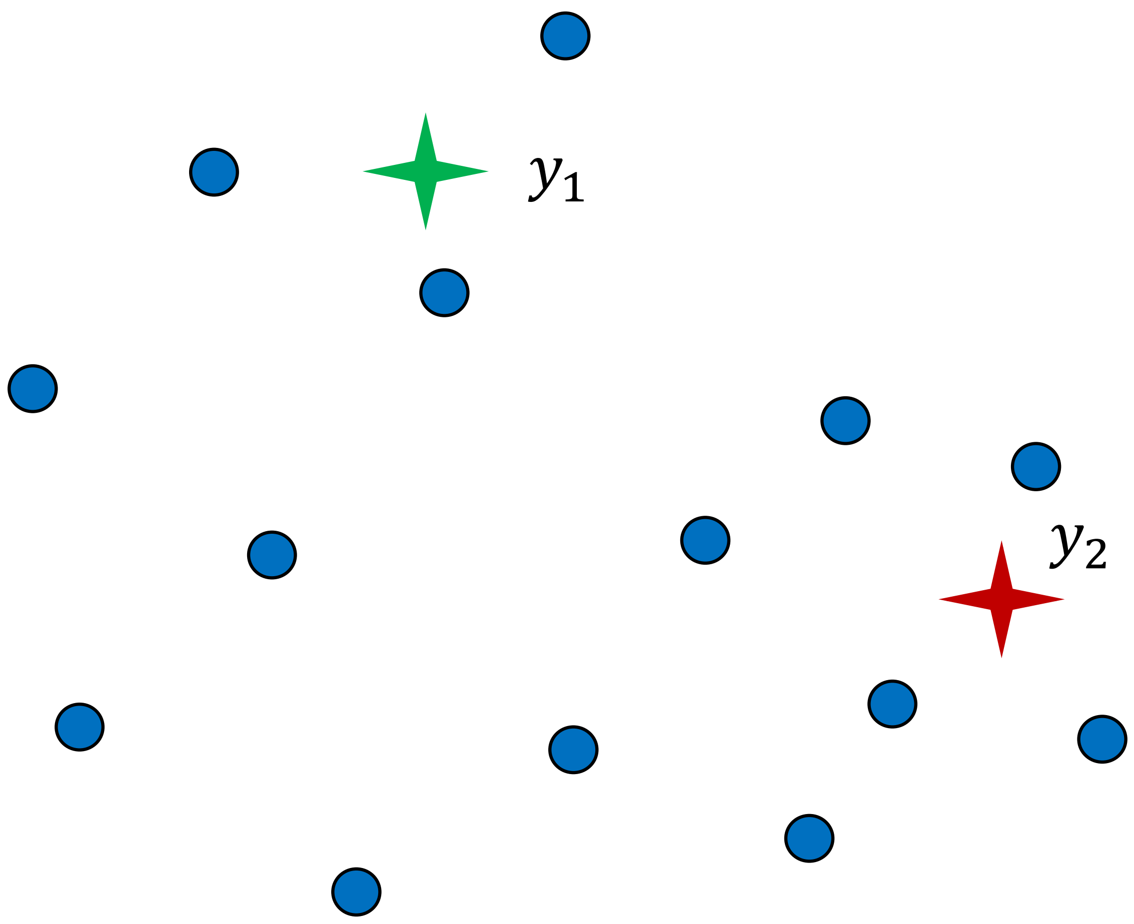

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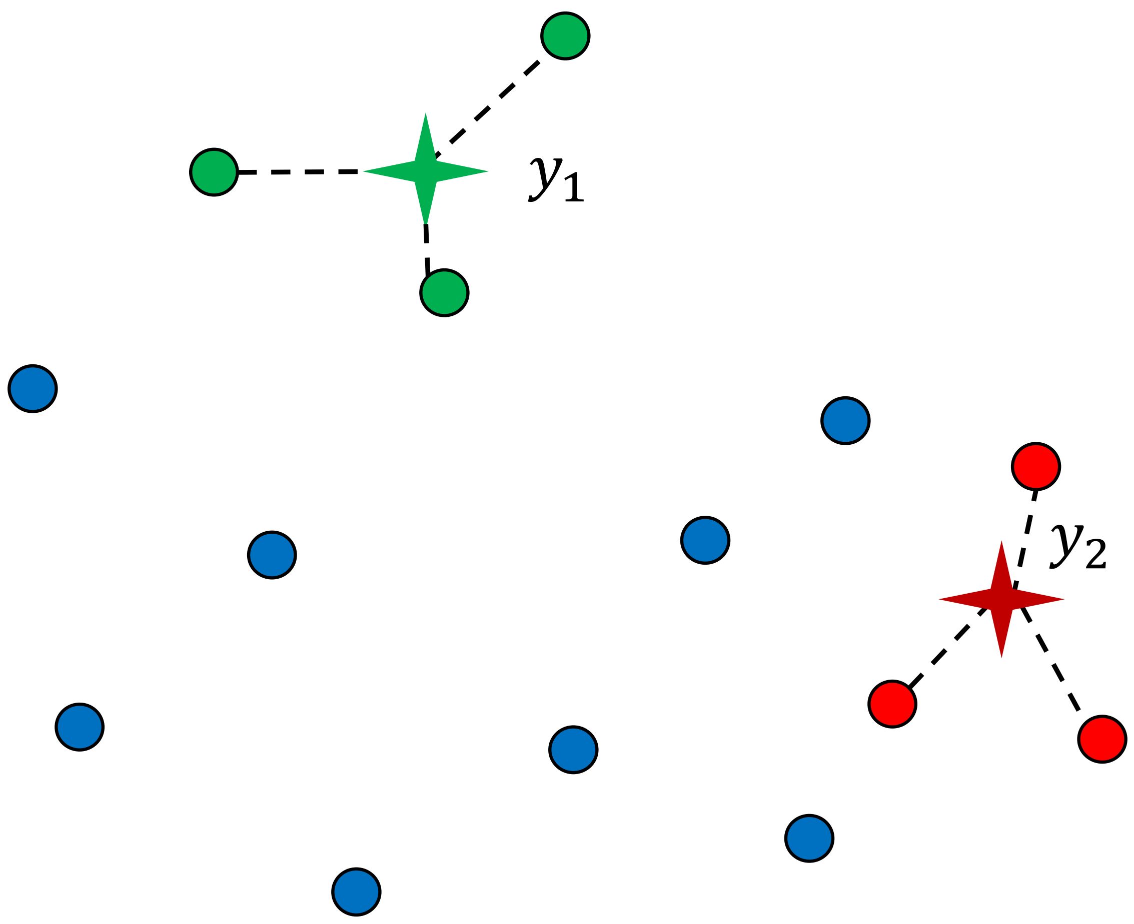

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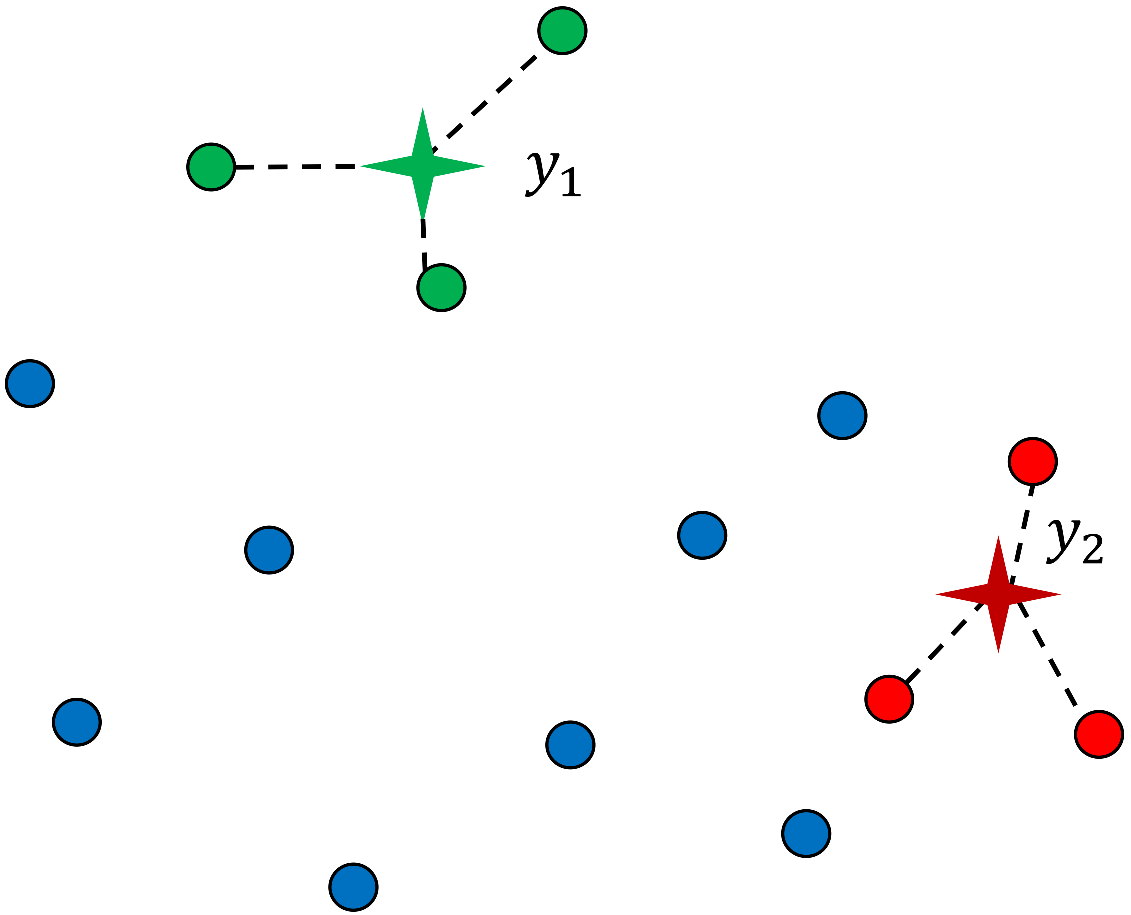

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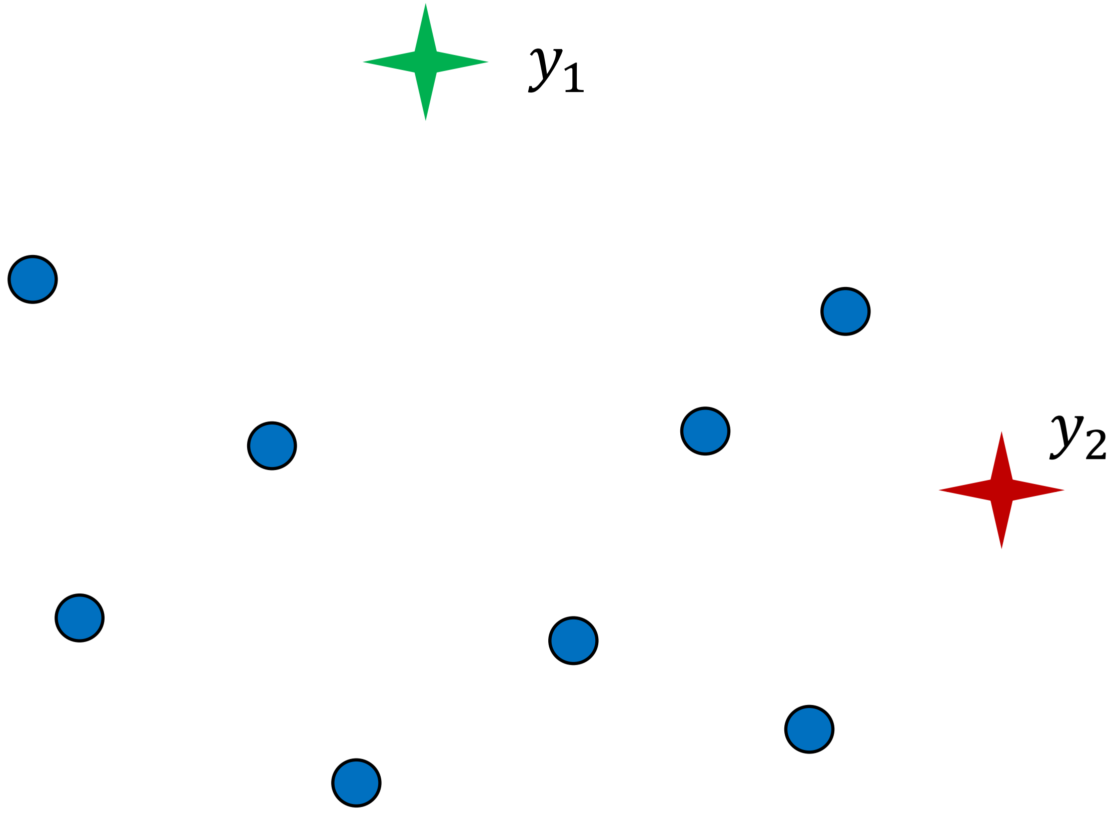

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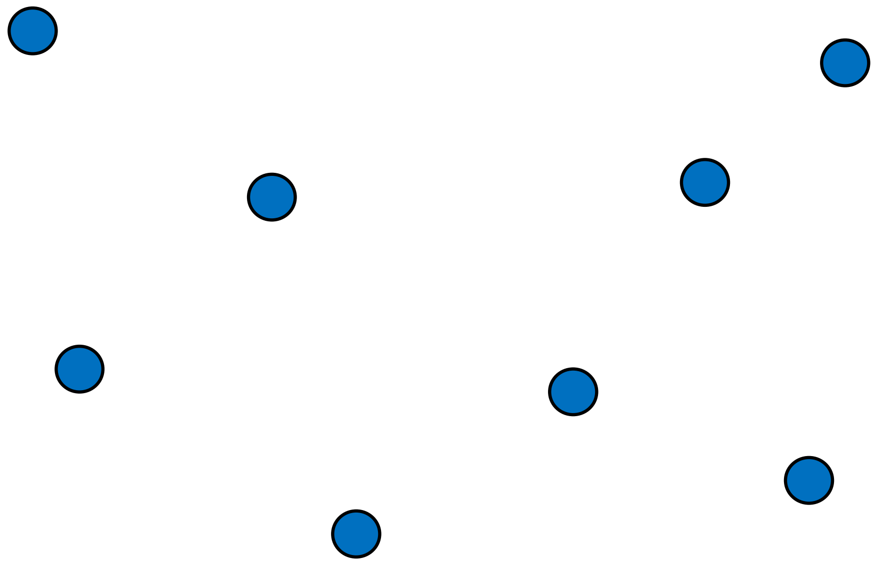

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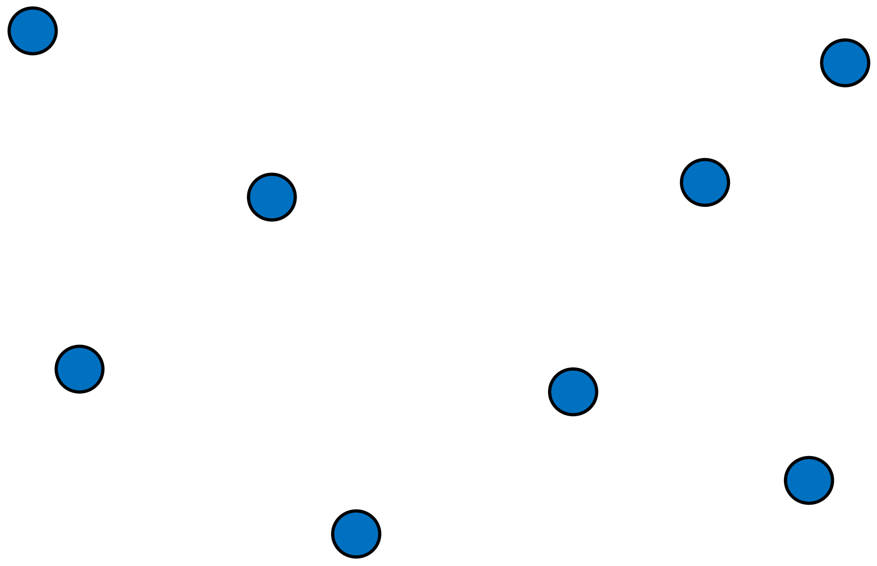

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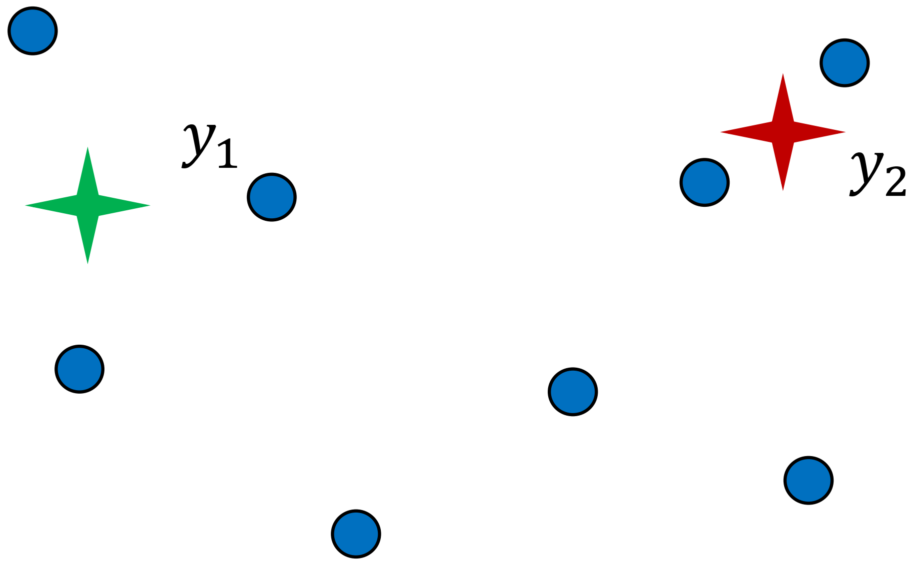

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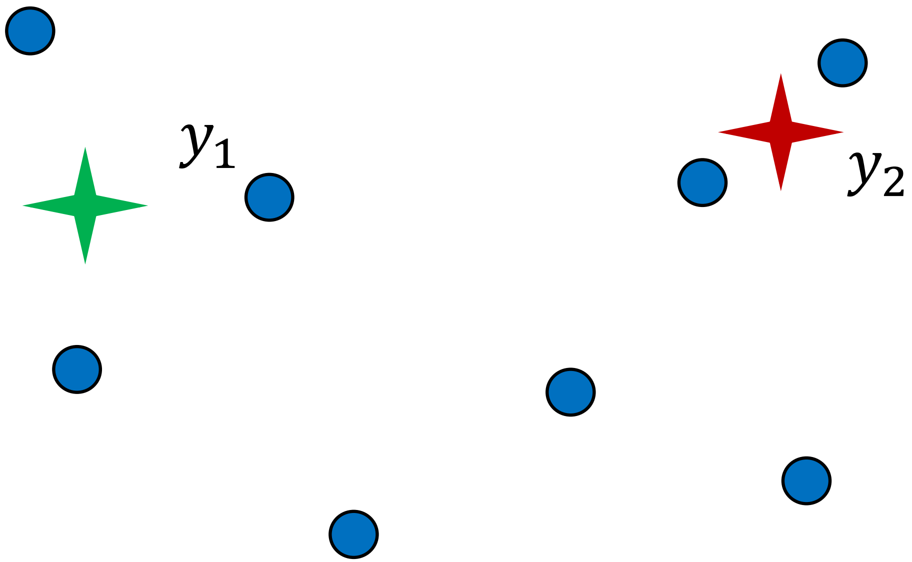

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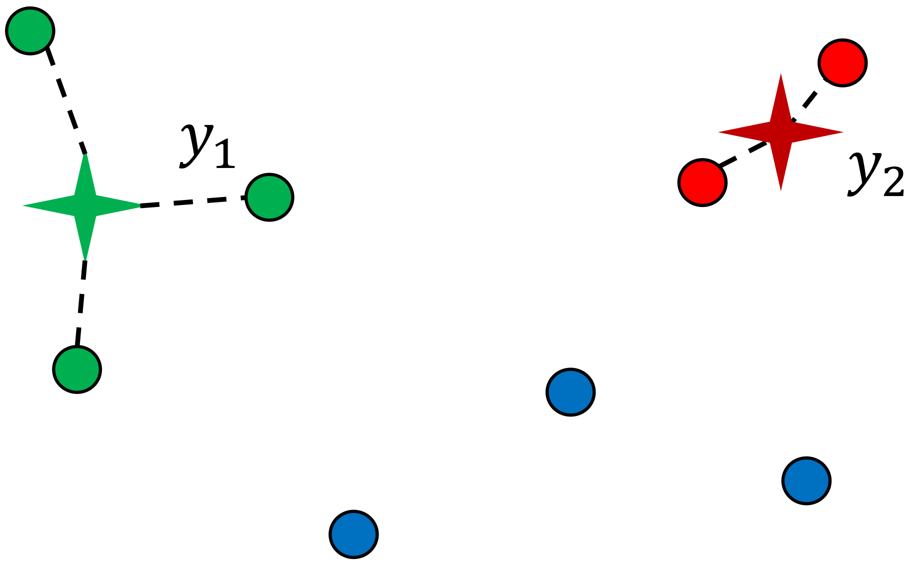

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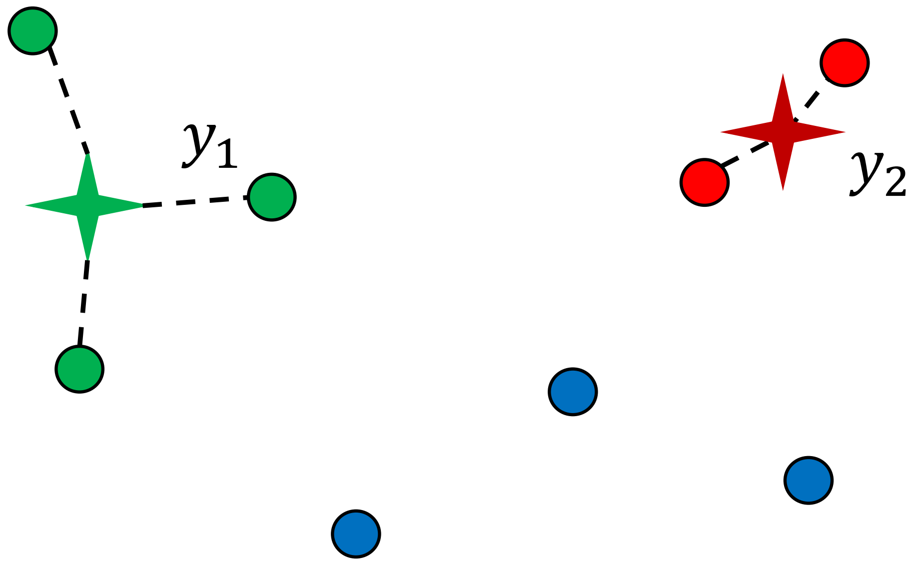

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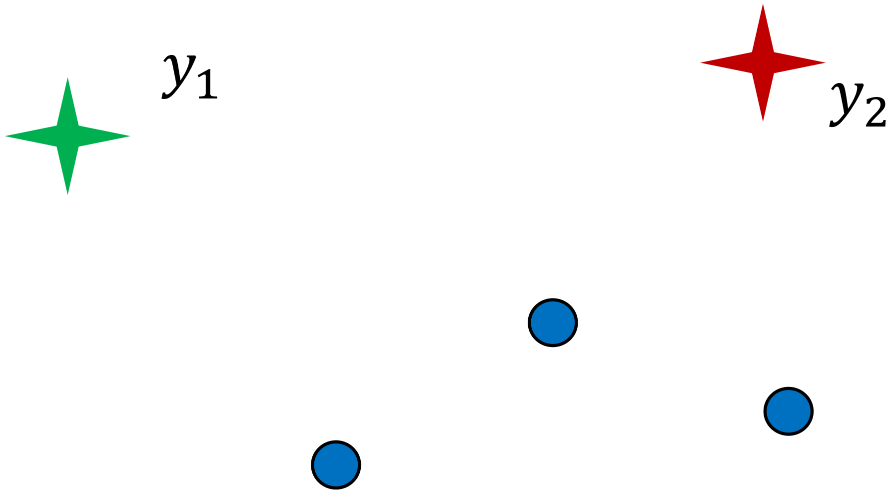

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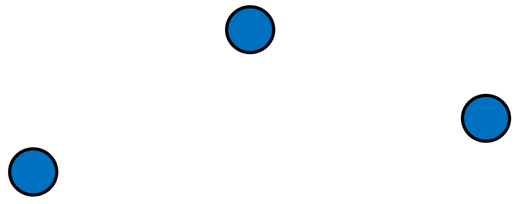
$\boldsymbol{Y = Y_1 \cup Y_2 \cup \cdots \cup Y_i}$



General Bi-creteria Algorithm

- **Claim:**

The algorithm $\boldsymbol{BICRETERIA(P, dist, \epsilon, \alpha, \beta)}$ returns an $\big(O(\alpha), O(\beta \log n)\big)$-$approximation.$

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- **Proof:**

Let Y^* be any set of k points in R^d

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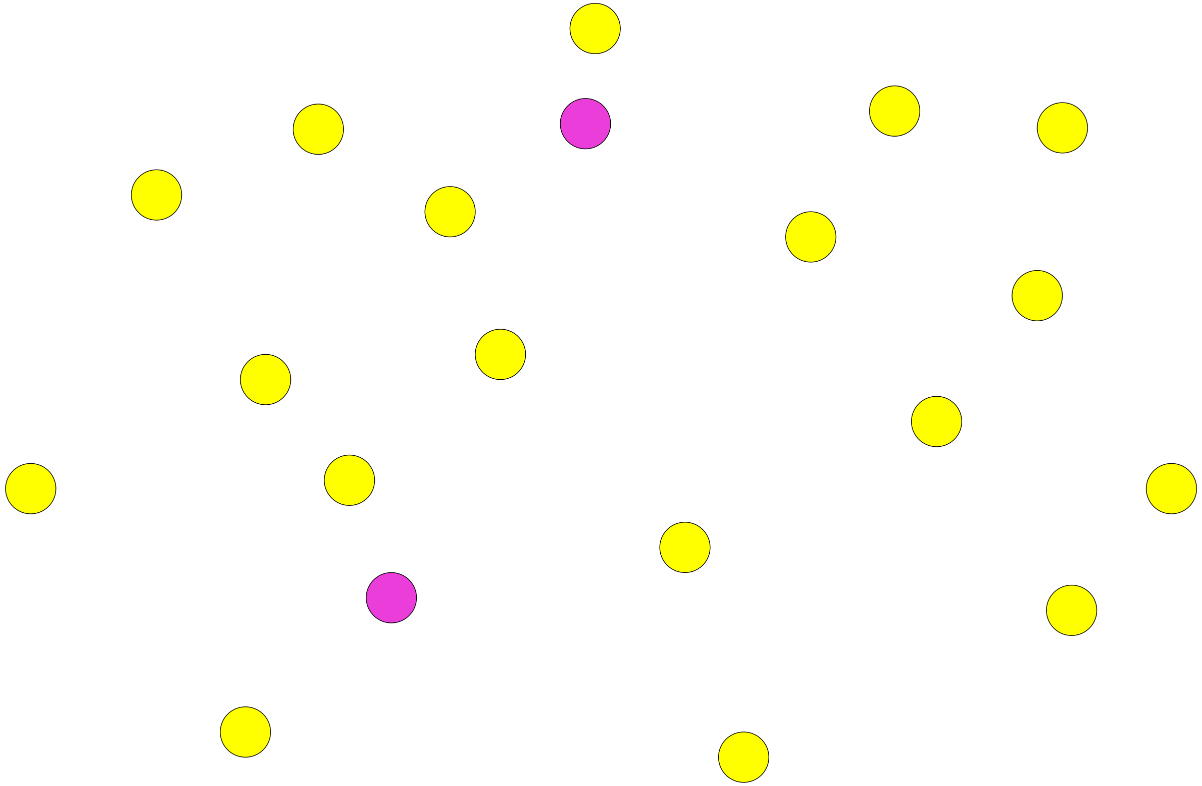

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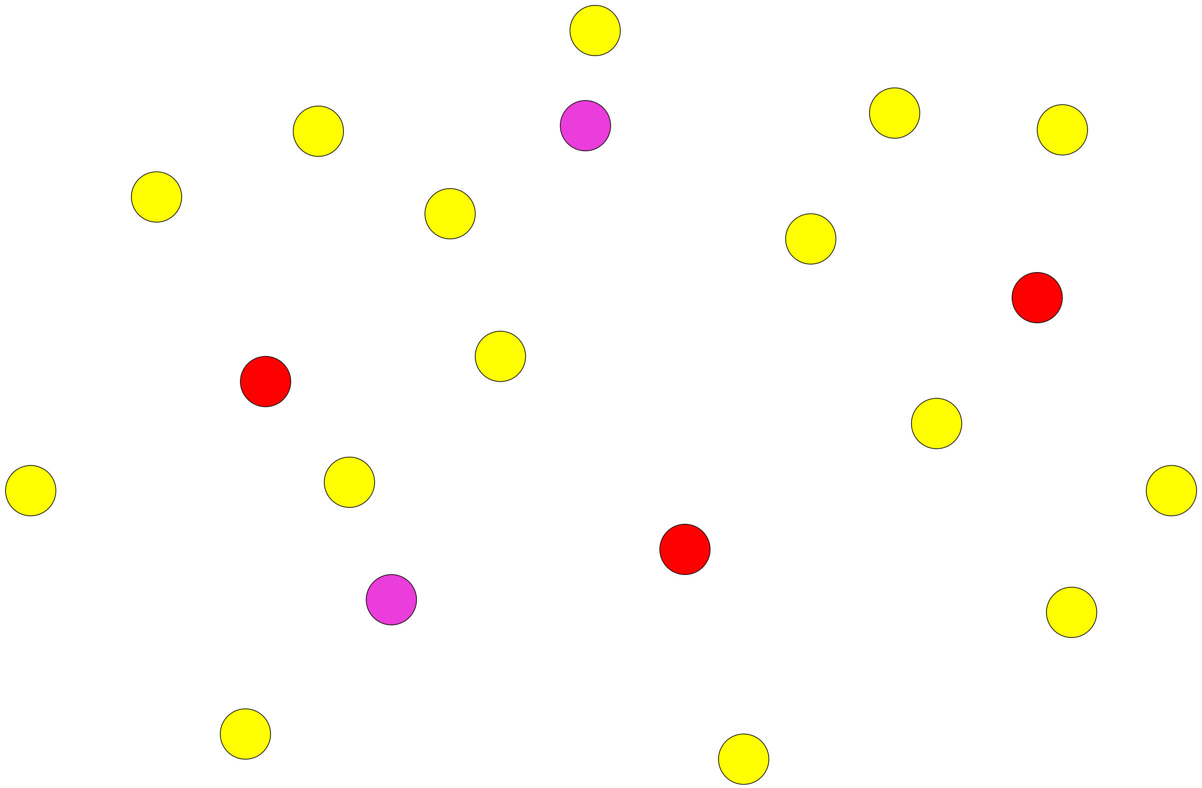
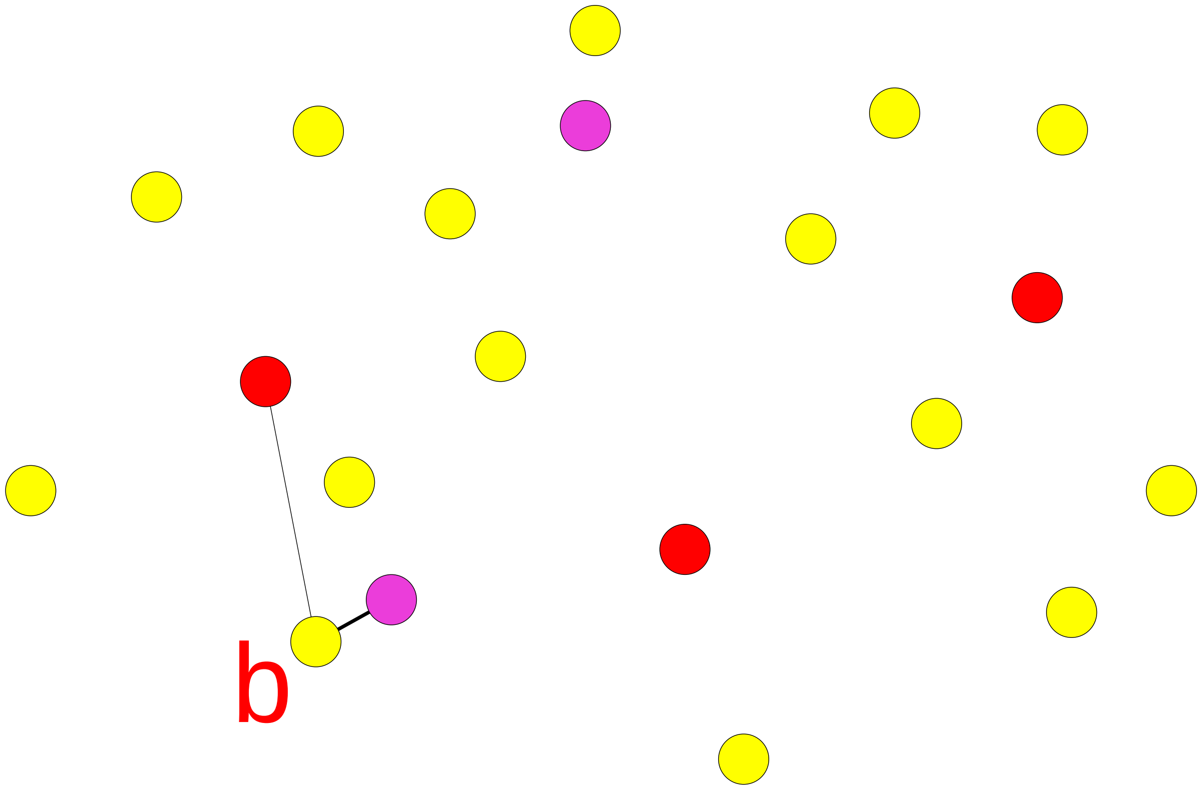


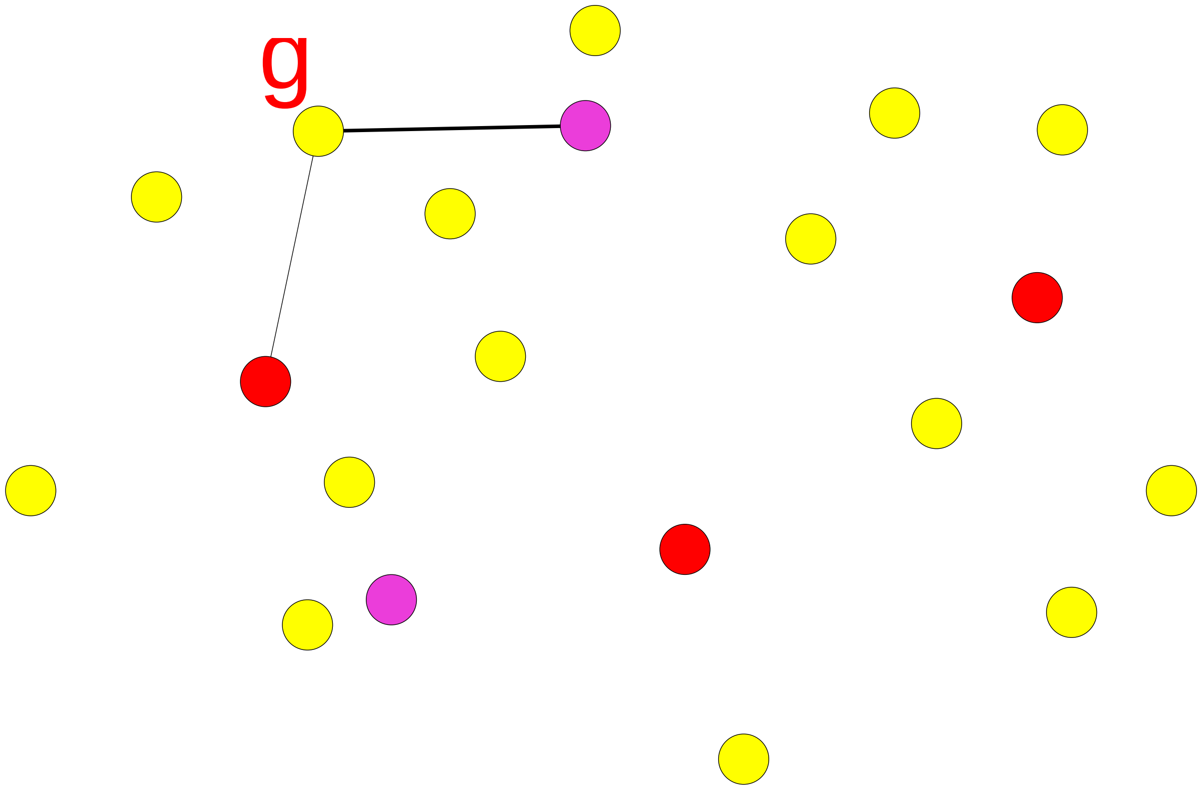
Consider Y_i that is constructed during the i^{th} iteration

A point $b \in P$ is bad for Y_i, if:



$$dist(b, Y_i) > 2 \cdot dist(b, Y^*)$$

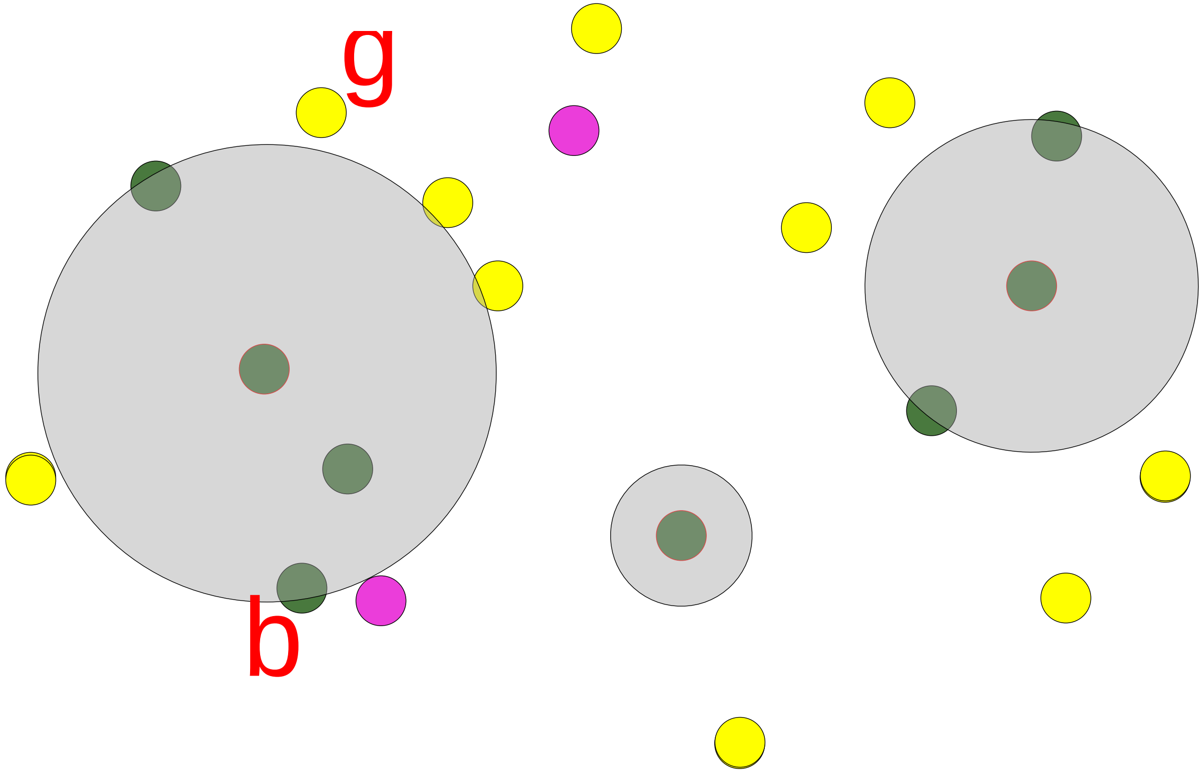
A point $g \in P$ is good for Y_i otherwise:



$$dist(g, Y_i) \leq 2 \cdot dist(b, Y^*)$$

Main Technical Theorem

We can map every bad point $b \in P_i$ to a distinct good point $g \in P_{i+1}$



$dist(b, Y) \leq dist(b, Y_i)$, because $Y_i \subseteq Y$.

Since $b \in P_i$ and $g \in P_{i+1}$:

$$dist(b, Y_i) \leq dist(g, Y_i)$$

Since g is good for Y_i :

$$dist(g, Y_i) \leq 2 \cdot dist(g, Y^*)$$

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Since g is good for Y_i :

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$$\Downarrow$$

$$dist(b, Y) \leq 2 \cdot dist(g, Y^*)$$

Bi-Criteria for k-Median

$$\sum_{p \in P} dist(p, Y) = \sum_{g} dist(g, Y) + \sum_{b} dist(b, Y)$$

$$\leq \sum_{g} 2 \cdot dist(g, Y^*) + \sum_{g} 2 \cdot dist(g, Y^*)$$

$$\leq \sum_{g} 4 \cdot dist(g, Y^*)$$

Proof of the Technical Theorem

- The number of bad points is at most:

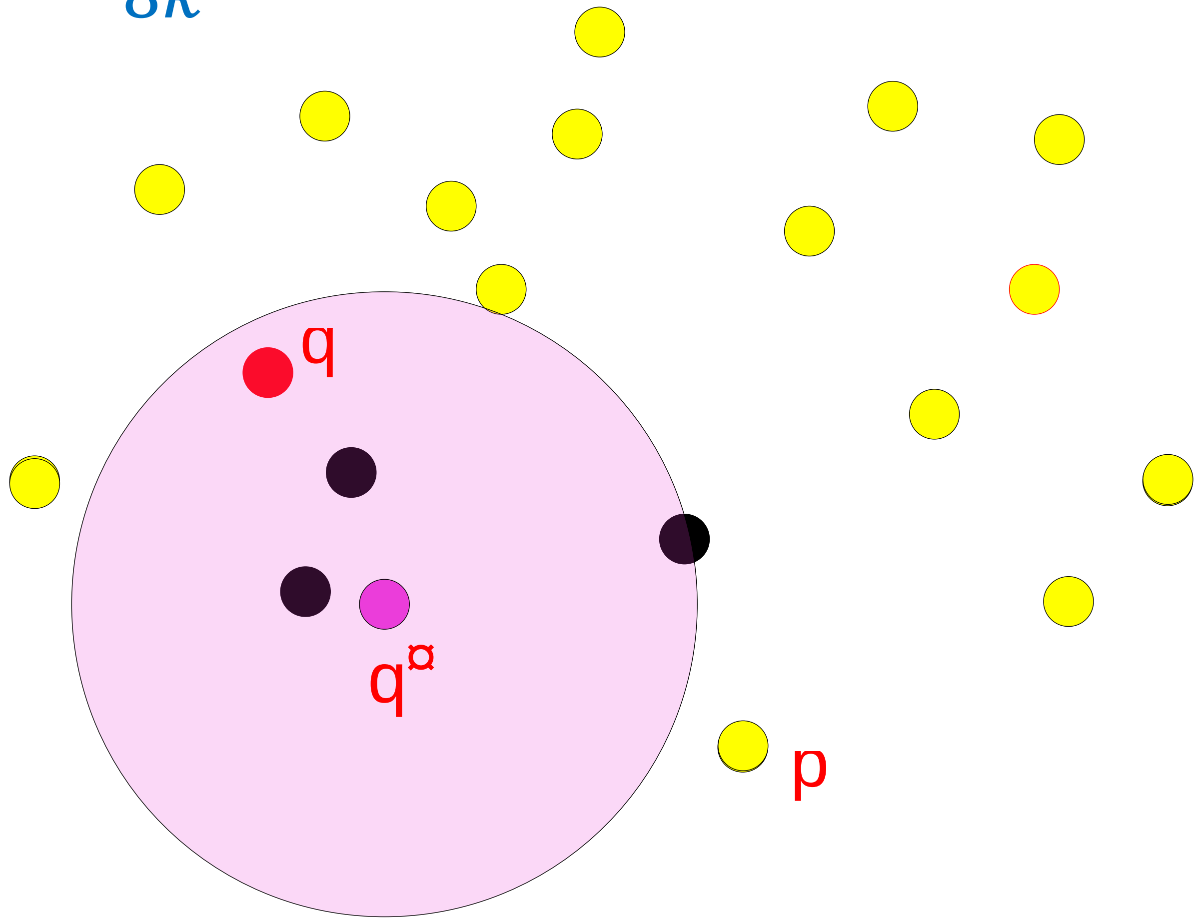
$$|B| = \frac{|Y_i|}{8}$$

- $|Y_{i+1}| = \frac{|Y_i|}{2}$

The number of good points in Y_{i+1} is at most:

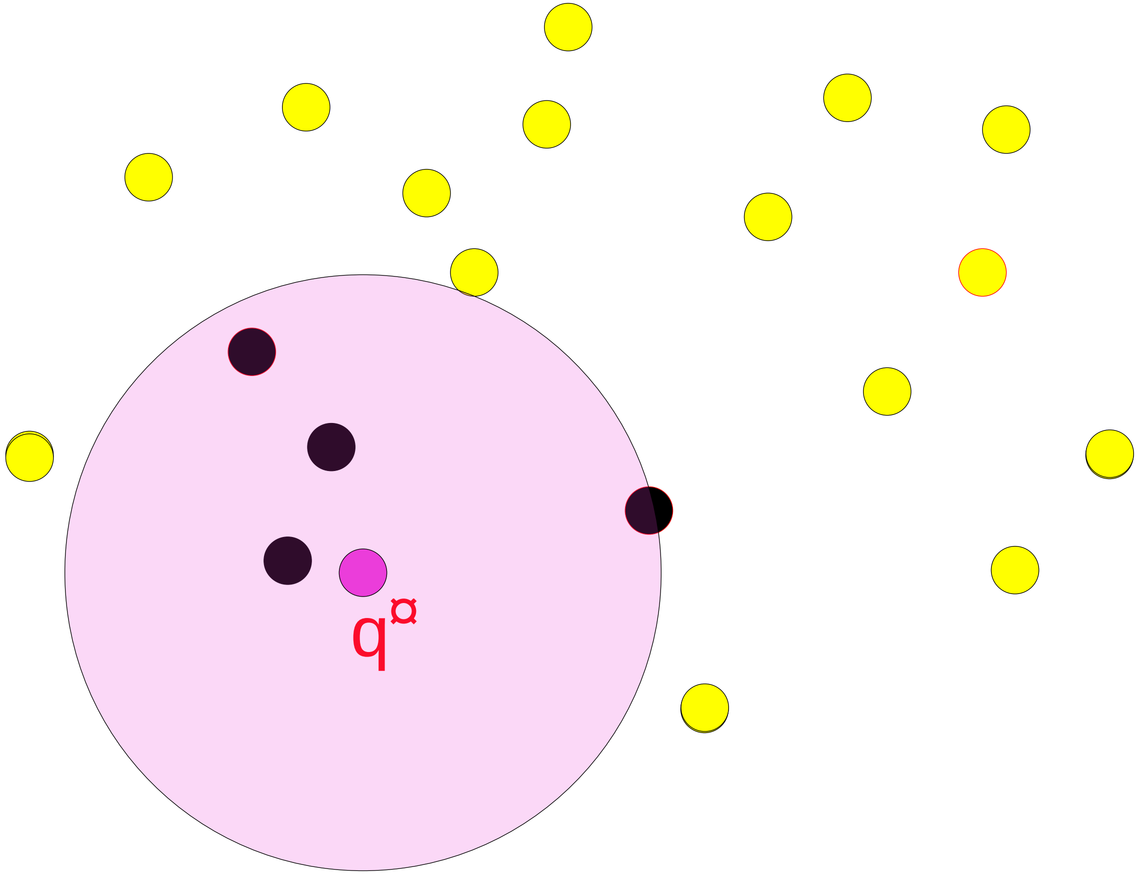
$$|Y_{i+1}| - |B| \geq \frac{|Y_i|}{2} - \frac{|Y_i|}{8} \geq |\mathrm{B}|$$

Claim: Only $B_0 = \frac{|Y_i|}{8k}$ points are bad for $q \in Y_i$



$$dist(p,q) \leq 2 \cdot dist(p,q^*)$$

B_0: the $\frac{|Y_i|}{8k}$ closest points to q^*



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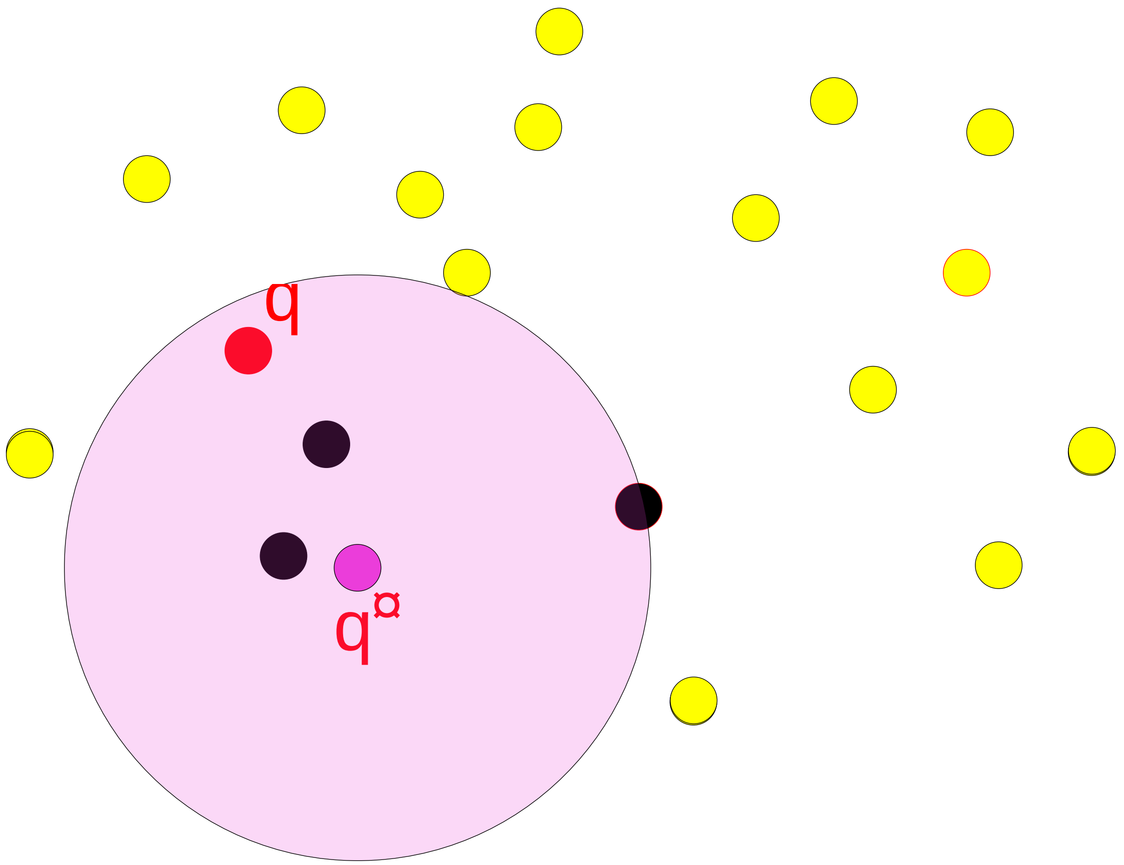
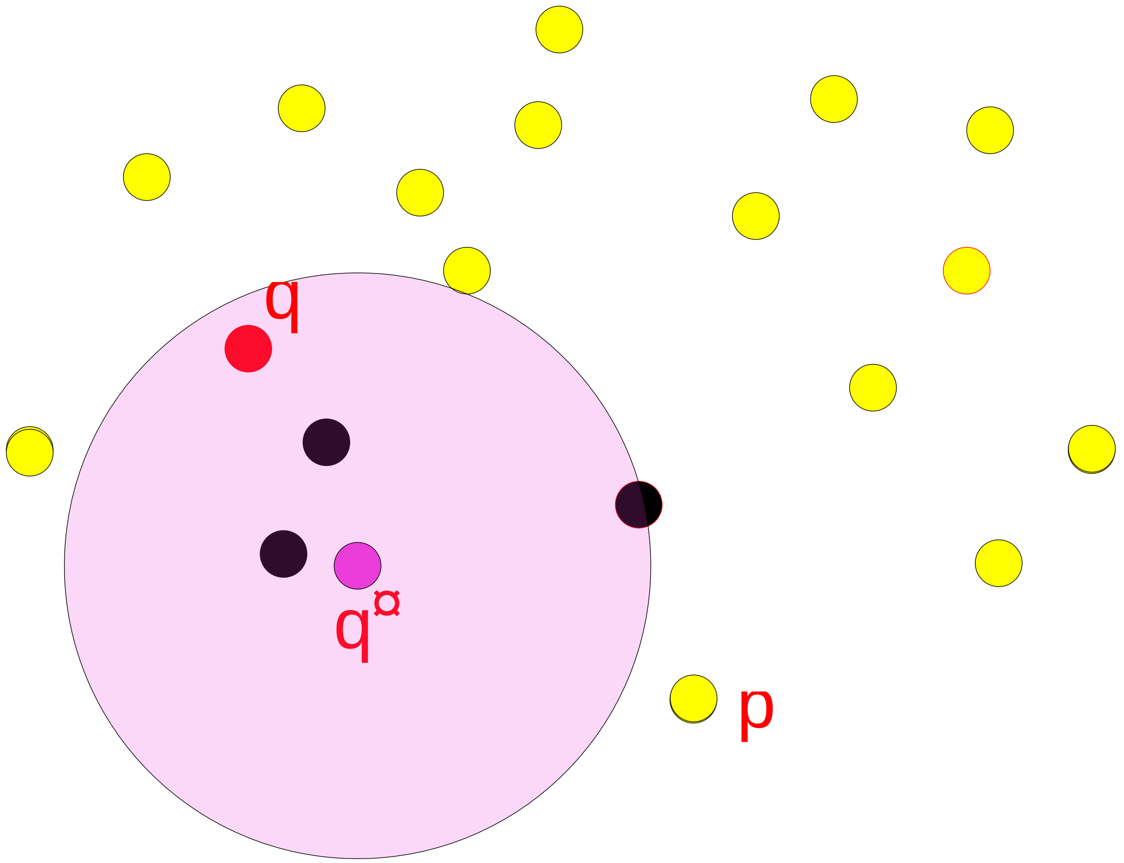


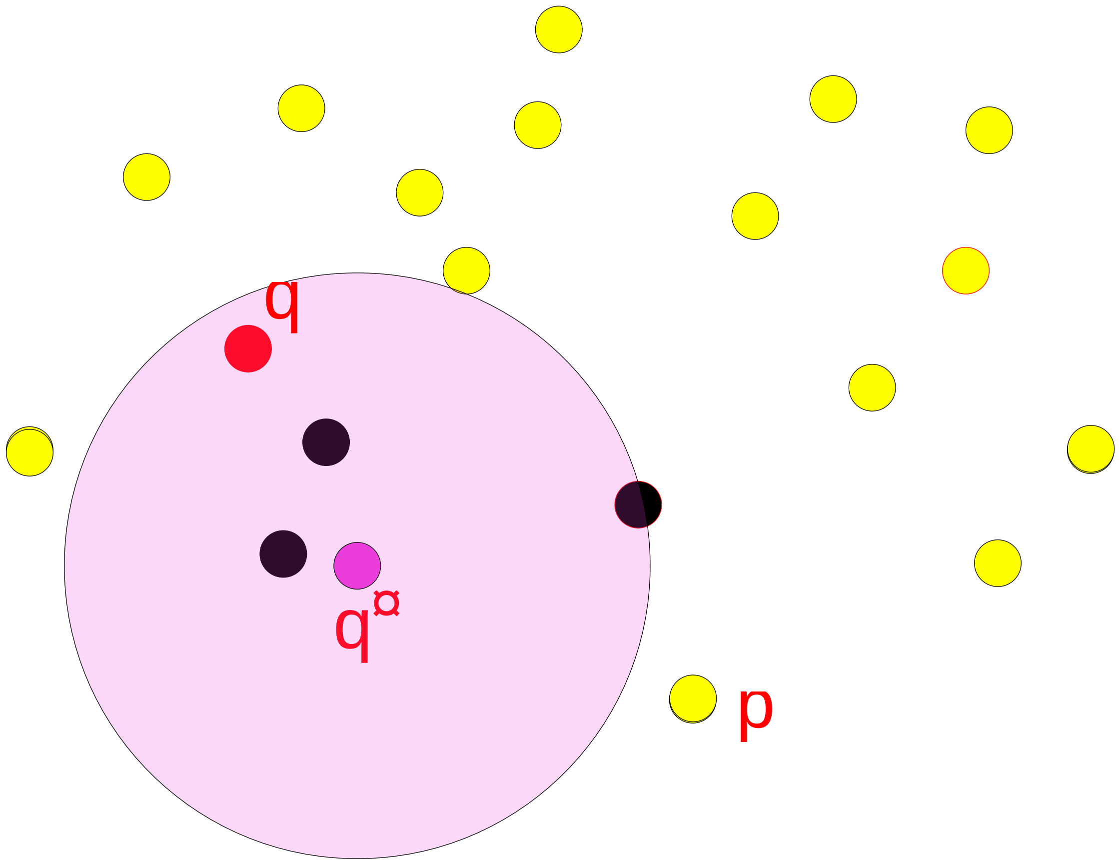
B_0 contains $q \in Y_i$ $(\frac{1}{8k} - net)$

For every yellow point $p \in P \backslash B_0$:



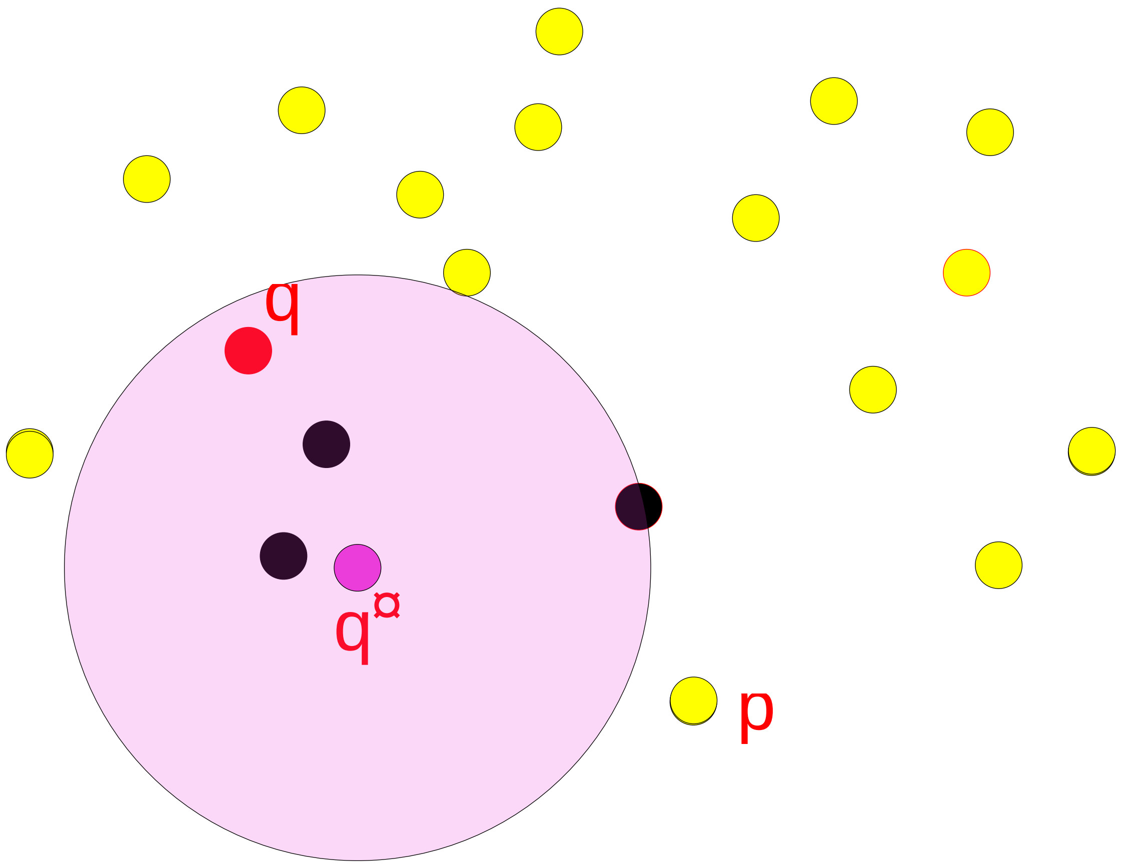
$$dist(p,q) \leq dist(p,q^*) + dist(q^*,q)$$
$$\leq 2 \cdot dist(p,q^*)$$

All the yellow points are good for Y_i



$$dist(p,q) \leq 2 \cdot dist(p,q^*)$$

Only the black points B_0 are bad for Y_i



$$B_0 = \frac{|Y_i|}{8k}$$